



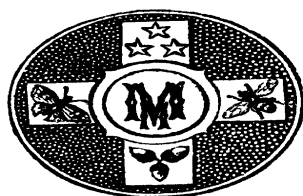




A GEOMETRICAL TREATISE

ON

C O N I C   S E C T I O N S.



A  
GEOMETRICAL TREATISE  
ON  
CONIC SECTIONS.

WITH A COPIOUS COLLECTION OF EXAMPLES,  
EMBODYING EVERY QUESTION WHICH HAS BEEN PROPOSED IN  
THE SENATE HOUSE AT CAMBRIDGE.

*For the Use of Schools and Students in the Universities.*

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## PREFACE

TO THE SECOND EDITION.

THE limits which I have assigned to myself both in this and the former edition, and which I have endeavoured not to transgress either in excess or defect, are the requirements of the Senate-House Examination at Cambridge. After a very careful revisal, I have come to the conclusion that the Propositions which I have given are fully sufficient for the solution of all such questions as can be conveniently treated geometrically. At the same time there is no proposition which can be omitted without the risk of some important point being overlooked. Though the alterations in the present edition are not extensive, they are important; one or two of the demonstrations have been much shortened and simplified, a footnote has been inserted on the first proposition both of the Ellipse and of the Hyperbola, where the text seemed obscure; a fresh proposition has been introduced on the circle of curvature in the Parabola and Ellipse, which is also applicable to the



Hyperbola ; the letters of some of the figures have been changed to avoid the confusion which arose from the multiplication of dashes and suffixes ; many corollaries have been added with a view of calling attention to points which had been passed over without notice. Finally, 130 fresh problems, selected from Examination Papers of recent dates, have been appended, which it is believed will afford the student all the opportunities for practising himself in this subject which he can desire.

The solutions of the problems contained in the former edition have been published in such a form as to be of service not only to the teacher, but also to the pupil. The additional problems, which are not referred to in the solutions, are placed separately at the end of the book. With the solutions the book now consists of three parts—the book-work, problems solved by way of illustration, and unsolved problems for the independent exercise of the student.

W. II. DREW.

BLACKHEATH PROPRIETARY SCHOOL,

*February 18th, 1862.*

## P R E F A C E.

THE importance of studying the properties of the Conic Sections by Geometrical methods before entering upon the algebraic theory cannot, I think, be too much insisted upon, whether regarded as an introduction to the study of Newton, or with a view to a more complete understanding subsequently of the principles of Analytical Geometry.

The advantage of observing this order is now so fully recognised at Cambridge, that during the first three days of the Examination for Honours the geometrical method is alone admissible.

At the same time I believe that this mode of treating the subject has not received in Schools the attention which it deserves, and I cannot but think that this is in a great degree owing to the want of a book altogether suited for the purpose.

In the following pages, which were originally compiled for the use of my own pupils at Blackheath,

I have aimed at supplying this deficiency, and I have endeavoured to place the subject before the student in such a form, that, after mastering the elements of Euclid, he may find it an easy and interesting continuation of his geometrical studies.

With a view also of rendering the work a complete Manual of what is required at Cambridge, I have either embodied into the text, or inserted among the examples, every book-work question, problem, and rider, which has been proposed in the Senate House up to the present time.

The principal points in which the present Treatise will be found to differ from those now in use are the following :—

(1.) The three Conic Sections are defined in a uniform manner.

(2.) The use of the Second Book of Euclid is avoided, *as much as possible*, as having a tendency to lead to algebraical methods of reasoning.

(3.) The properties of the ellipse and hyperbola which depend upon the directrix are fully given, and the analogy between these curves and the parabola is thus maintained.

(4.) A method of proving the fundamental property of the tangents has been adopted, which, while it presents the idea of a limit in its simplest form, is

applicable, word for word, to each of the three curves.

(5.) With a view to simplification, several of the demonstrations have been much modified from the form in which they are usually presented, among which I may particularly mention those propositions which relate to the properties of conjugate diameters in the ellipse, and of the asymptotes in the hyperbola, and the proof that  $QV^2 = 4SP \cdot PT$  in the parabola; while many important theorems admitting of an easy geometrical solution are introduced.

(6.) A full discussion is given in a distinct chapter of the Sections of the Cone. Figures are drawn representing the position of the foci and of the directrices of the sections in every case.

W. H. DREW.

BLACKHEATH PROPRIETARY SCHOOL,

*March 19th, 1857.*



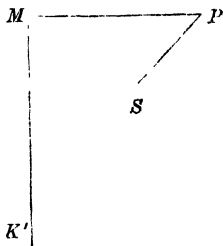
# CONIC SECTIONS.

## INTRODUCTION.

1. DEF. The curve traced out by a point, which moves in such a manner that its distance from a given fixed point continually bears the same ratio to its distance from a given fixed line, is called a *Conic Section*.

The fixed point is called the *Focus*, and the fixed line the *Directrix*.

Thus if  $S$  be the focus, and  $KK'$  the directrix, and  $P$  a point from which  $PM$  is drawn at right angles to the directrix, the curve traced out by  $P$  will be a *Conic Section*, provided  $P$  move in such manner that  $SP$  always bears the same ratio to  $PM$ .



(1.) When the distance from the fixed point is equal to the distance from the fixed line, that is, when  $SP$  is equal to  $PM$ , the *Conic Section* is called a *Parabola*.

(2.) When the distance from the fixed point is less than the distance from the fixed line, that is, when the ratio which

$SP$  bears to  $PM$  is less than unity, the *Conic Section* is called an *Ellipse*.

(3.) When the distance from the fixed point is greater than the distance from the fixed line, that is, when the ratio which  $SP$  bears to  $PM$  is greater than unity, the *Conic Section* is called an *Hyperbola*.

2. The reason of the term *Conic Sections* being applied to these curves is that, when a *Cone* is intersected by a plane surface, the boundary of the section so formed will, *in general*, be one or other of these curves.

I purpose to investigate the properties of the *Conic Sections* from the definitions given above, and afterwards to show in what manner a *Cone* must be divided by a plane in order that the curve of intersection may be a *Parabola*, *Ellipse*, or *Hyperbola*.

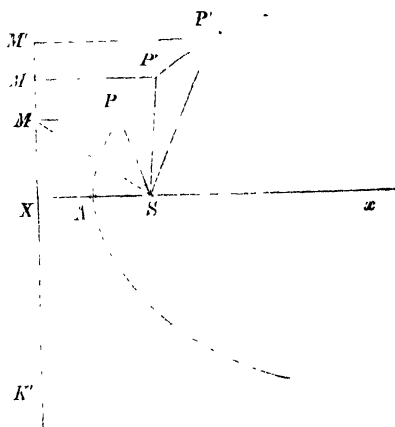


# CHAPTER I.

## THE PARABOLA.

### PROP. I.

3. THE focus and directrix of a parabola being given, to find any number of points on the curve.



Let  $S$  be the focus, and  $KK'$  the directrix.

Draw  $XSx$  at right angles to the directrix, and bisect the line  $SX$  in  $A$ ; then

$$\text{since } AS = AX,$$

$\therefore A$  is a point on the curve.

The point  $A$  is called the *Vertex*, and the line  $Ax$ , with respect to which the curve is evidently symmetrical, is called the *Axis*.



On the directrix take any point  $M$ ; join  $SM$ ; and draw  $MP$  at right angles to the directrix.

At the focus  $S$  make the angle  $MSP$  equal to the angle  $SMP$ ; then

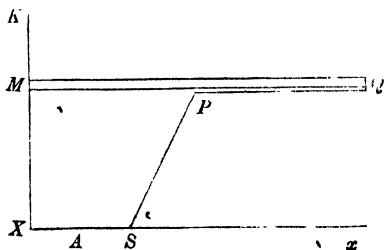
$$SP = PM,$$

$\therefore P$  is a point on the curve.

So by taking any number of points,  $M'$ ,  $M''$ , on the directrix, we may obtain as many points,  $P'$ ,  $P''$ , on the curve as we please, and the line which passes through  $A$  and all these points will be the parabola whose focus is  $S$  and directrix  $KX$ .

COR. 1. As  $M$  is taken further away from the point  $X$ , the line  $SM$  and the angles  $SMP$ ,  $MSP$ , and, consequently, the lines  $SP$  and  $PM$ , continually increase. Hence, since  $XM$  and  $MP$  increase together, the curve recedes at the same time both from the axis and directrix; and since the angle  $SMP$  can never exceed a right angle, and the lines  $SP$  and  $MP$  will therefore always meet, it is evident that there is no limit to the distance to which the curve may extend on both sides of the axis.

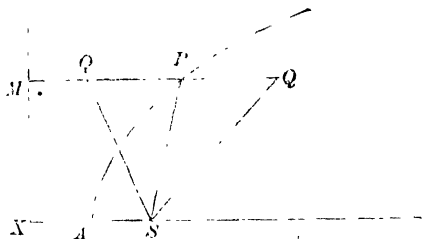
COR. 2. The parabola may be described practically in the following manner.



Let  $S$  be the focus and  $KX$  be the directrix; and let a rigid bar  $QM$ , having a string of the same length as itself fastened at one end  $Q$ , be made to slide parallel to the axis with the other end  $M$  on the directrix; then if the other end of the string be fastened at the focus, and the string be kept stretched by means of the point of a pencil at  $P$ , in contact with the bar, since  $SP$  will always be equal to  $PM$ , it is evident that the point  $P$  will trace out the parabola.

PROP. II.

4. The distance of any point inside the parabola from the focus is less than its distance from the directrix; and the distance of any point outside the parabola from the focus is greater than its distance from the directrix.



(1.) Let  $Q$  be a point inside the parabola.

Draw  $QM$  at right angles to the directrix, meeting the parabola in  $P$ ; join  $SP$ ; then

since  $SP = PM$ ,

$\therefore SP$  and  $PQ = QM$ .

But  $SP$  and  $PQ > SQ$ .

$$\therefore QM > SQ.$$

(2.) Let  $Q$  be a point outside the parabola.

Draw  $MQ$  at right angles to the directrix, and produce it to meet the parabola in  $P$ ; join  $SP$ ; then

since  $SQ$  and  $QP > SP$ ,

and  $SP = PM$ ,

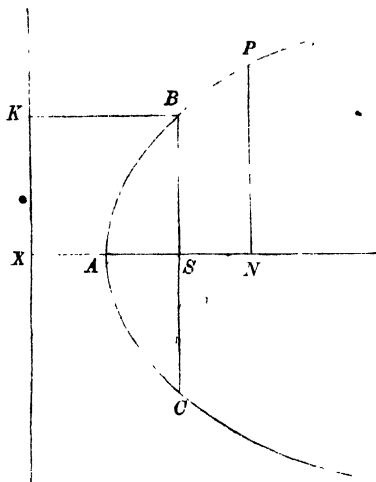
$$\therefore SQ \text{ and } QP > PM,$$
$$\therefore SQ > QM.$$

COR. Conversely a point will be inside or outside the parabola according as its distance from the focus is less or greater than its distance from the directrix.

5. DEF. The line  $PN$  (see *fig. Prop. III.*) drawn at right angles to the axis from the point  $P$  in the curve is called the *Ordinate* of the point  $P$ , and the line  $AN$  the *Abscissa*. The double ordinate  $BC$  drawn through the focus, and terminated both ways by the curve, is called the *Latus Rectum*.

## PROP. III.

The Latus Rectum  $BC = 4AS$ .



Draw  $BK$  at right angles to the directrix.

Then  $SB = BK = SX = 2AS$ ,

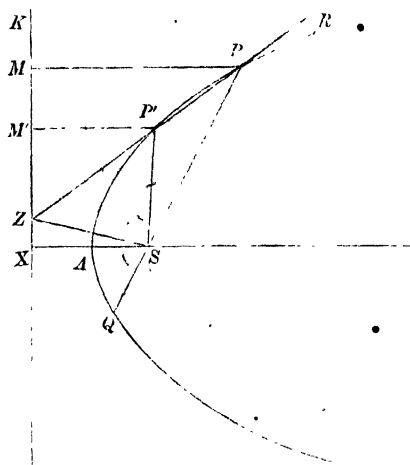
$\therefore BC = 4AS$ .

6. DEF. If a point  $P'$  be taken on the parabola (see fig. *Prop. IV.*) near to  $P$ , and  $PP'$  be joined, the line  $PP'$  produced, in the limiting position which it assumes when  $P'$  is made to approach indefinitely near to  $P$ , is called the *Tangent* to the parabola at the point  $P$ .

## PROP. IV.

• If the tangent to the parabola at any point  $P$  intersect the directrix in the point  $Z$ ; then  $SZ$  will be at right angles to  $SP$ .

Let  $P'$  be a point on the parabola near to  $P$ .



Draw the chord  $PP'$ , and produce it to meet the directrix in  $Z$ ; join  $SZ$ .

Draw  $PM$ ,  $P'M'$  at right angles to the directrix; join  $SP$ ,  $SP'$ ; and produce  $PS$  to meet the parabola in  $Q$ .

Then, since the triangles  $ZMP$ ,  $ZM'P'$  are similar,

$$\therefore ZP : ZP' :: MP : M'P',$$

$$:: SP : SP',$$

$\therefore SZ$  bisects the angle  $P'SQ$ . (*Euclid*, VI. *Prop.* A.)

Now when  $P'$  is indefinitely near to  $P$ , and  $PP'$  becomes the tangent at the point  $P$ , the angle  $PSP'$  becomes indefinitely small, while the angle  $QSP'$  approaches two right angles, and therefore the angle  $P'SZ$ , which is half of the angle  $P'SQ$ , becomes ultimately a right angle.

Hence, when  $PZ$  is the tangent,

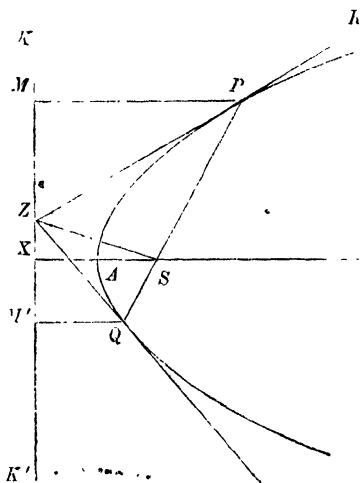
the angle  $ZSP$  is a right angle,

or  $SZ$  is perpendicular to  $SP$ .

COR. Conversely, if  $SZ$  be drawn at right angles to  $SP$ , meeting the directrix in  $Z$ , and  $PZ$  be joined,  $PZ$  will be a tangent at  $P$ .

## PROP. V.

7. The tangent at any point  $P$  of a parabola bisects the angle between the focal distance  $SP$ , and the perpendicular  $PM$  on the directrix.



Let the tangent at  $P$  meet the directrix in the point  $Z$ ; join  $SZ$ ; then since the angle  $ZSP$  is a right angle, (*Prop. IV.*)

$$\therefore ZS^2 + SP^2 = PZ^2.$$

$$\text{Also } ZM^2 + MP^2 = PZ^2,$$

$$\therefore ZS^2 + SP^2 = ZM^2 + MP^2.$$

$$\text{But } SP = PM,$$

$$\therefore ZS = ZM.$$

Now in the triangles  $ZPS$ ,  $ZPM$ ,

$$\therefore ZP, PS = ZP, PM, \text{ each to each.}$$

$$\text{and } ZS = ZM,$$

$$\therefore \text{the angle } SPZ = \text{the angle } MPZ;$$

$$\text{or } PZ \text{ bisects the angle } SPM.$$

COR. 1. If  $ZP$  be produced to  $R$ , then the angle  $SPR =$  the angle  $MPR$ .

COR. 2. It is evident that the tangent at the vertex  $A$  is perpendicular to the axis.

#### PROP. VI.

8. The tangents at the extremities of a focal chord intersect at right angles in the directrix.

Let  $PSQ$  be a focal chord, and let the tangent at  $P$  meet the directrix in  $Z$ .

Join  $SZ$ ; then

the angle  $ZSP$  is a right angle, (*Prop. IV.*)

and  $\therefore$  also the angle  $ZSQ$  is a right angle,

$\therefore ZQ$  is the tangent at  $Q$ , (*Prop. IV. Cor.*)

or the tangents at the extremities of the focal chord  $PSQ$  intersect in the directrix.

Again, draw  $PM$ ,  $QM'$  at right angles to the directrix; then

since  $MP$ ,  $PZ = SP$ ,  $PZ$ , each to each,

and the angle  $MPZ =$  the angle  $SPZ$ ,

$\therefore$  the angle  $MZP =$  the angle  $SZP$ ,

$\therefore$  the angle  $SZP$  is half of the angle  $SZM$ .

So the angle  $SZQ$  is half of the angle  $SZM'$ ,

$\therefore$  the angle  $PZQ$  is half of the two  $SZM$  and  $SZM'$ .

But the angles  $SZM$  and  $SZM' =$  two right angles;

$\therefore$  the angle  $PZQ$  is a right angle,

or the tangents at the extremities of a focal chord intersect at right angles in the directrix.



Since the angle  $SPG$  = the complement of the angle  $SPT$ ,  
 and the angle  $S~~PG~~$  = the complement of the angle  $STP$ ,  
 and also the angle  $SPT$  = the angle  $STP$ , (*Prop. VII.*)

$\therefore$  the angle  $SPG$  = the angle  $SGP$ ,

$\therefore SG = SP$ .

But  $SP = PM = XN$ ,

$\therefore SG = XN$ .

Taking away the common part  $SN$ ,

the remainder  $NG = SX = 2AS$ .

#### PROP. IX.

11. If  $PN$  be an ordinate to the parabola at the point  $P$ ;  
 then  $PN^2 = 4AS \cdot AN$ .

Since  $TPG$  is a right angle, and  $PN$  perpendicular to  $TG$ ,

$\therefore PN$  is a mean proportional between  $TN$  and  $NG$ ;

or  $PN^2 = TN \cdot NG$ . (*Euclid, VI. 8 Cor.*)

But  $TN = 2AN$ , (*Prop. VII.*)

and  $NG = 2AS$ . (*Prop. VIII.*)

$\therefore PN^2 = 4AS \cdot AN$ .

#### PROP. X.

12. If the tangent at any point  $P$  intersect the tangent at the vertex in  $Y$ ,  $SY$  will bisect  $PT$  at right angles, and will be a mean proportional between  $SA$  and  $SP$ .

Draw  $PN$  at right angles to the axis; then

since  $AY$  is parallel to  $PN$ ,

$\therefore TY : YP :: TA : AN$ .

But  $AT = AN$ , (*Prop. VII.*)

$\therefore TY = PY$ ;

and  $\therefore SY, YP = SY, YT$ , each to each,

and  $SP = ST$ , (*Prop. VII.*)

$\therefore$  the angle  $SYP$  = the angle  $SYT$ ,

$\therefore SY$  is perpendicular to  $PT$ .



Again, since  $TYS$  is a right angle, and  $YA$  perpendicular to  $ST$ ,

$\therefore SY$  is a mean proportional between  $ST$  and  $SA$ ;

or  $SY^2 = ST \cdot SA$ : (*Euclid*, VI. 8 *Cor.*)

But  $ST = SE$ ; (*Prop.* VII.)

$\therefore SY^2 = SP \cdot SA$ .  $\cdot 4SY^2 = 4SP \cdot SA = 4SA \cdot SP$ .

COR. If  $PM$  be drawn at right angles to the directrix, and  $MY$  be joined, then

since  $SP, PY = MP, PY$ , each to each,

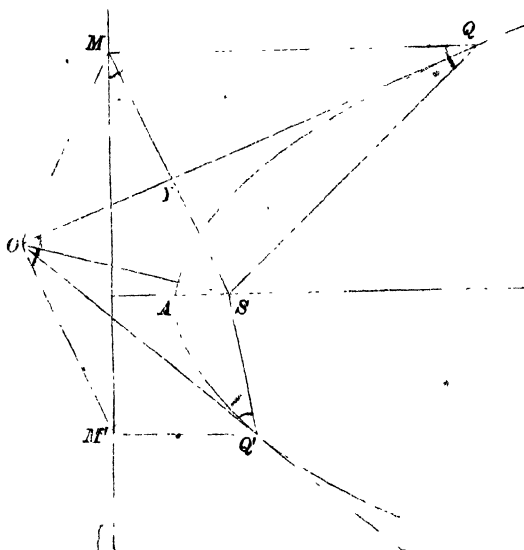
and the angle  $SPY =$  the angle  $MPY$ , (*Prop.* V.)

$\therefore$  the angle  $SYP =$  the angle  $MPY$ ,

$\therefore SY$  and  $YM$  are in the same straight line.

#### PROP. XI.

13. To draw a pair of tangents to a parabola from an external point.



## CONIC SECTIONS.

Let  $O$  be the given external point.

Join  $OS$ , and with centre  $O$  and radius  $OS$  describe a circle, cutting the directrix in  $M$  and  $M'$ , which it will always do, on whichever side of the directrix  $O$  is situated, since  $O$  is nearer to the directrix than to the focus. (*Prop. II.*)

Draw  $MQ$  and  $M'Q'$  parallel to the axis meeting the parabola in  $Q$  and  $Q'$ .

Join  $OQ$ ,  $OQ'$ ; these will be the tangents required.

Join  $SQ$  and  $SQ'$ ; then

$$\begin{aligned} \therefore OQ, QS &= OQ, QM, \text{ each to each,} \\ \text{and } OS &= OM, \end{aligned}$$

$$\therefore \text{ the angle } OQS = \text{ the angle } OQM,$$

$$\therefore OQ \text{ is the tangent at } Q. \quad (\text{Prop. V.})$$

$$\text{So } OQ' \text{ is the tangent at } Q'.$$

### PROP. XII.

14. If from a point  $O$  a pair of tangents  $OQ$  and  $OQ'$  be drawn to a parabola, the triangles  $OSQ$ ,  $OSQ'$  will be similar, and  $OS$  will be a mean proportional between  $SQ$  and  $SQ'$ .

Join  $SM$ , cutting  $OQ$  at right angles (*Prop. X. Cor.*) in the point  $Y$ ; then

$$\text{since the angle } SQO = \text{ the angle } MQO, \quad (\text{Prop. V.})$$

$$\text{and that the angle } MQO = \text{ the angle } SMM',$$

each of these angles being the complement of the angle  $QMY$ ,

$$\therefore \text{ the angle } SQO = \text{ the angle } SMM'.$$

But the angle  $SMM'$  at the circumference is half the angle  $SOM'$  at the centre, and is therefore equal to the angle  $SOQ'$ .

$$\therefore \text{ the angle } SQO = \text{ the angle } SOQ'.$$

$$\text{So the angle } SOQ = \text{ the angle } SQ'Q,$$

$$\therefore \text{ the remaining angle } OSQ = \text{ the remaining angle } OSQ'.$$

And therefore the triangle  $OSQ$  is similar to the triangle  $OSQ'$ ,

$$\therefore SQ : SO :: SO : SQ',$$

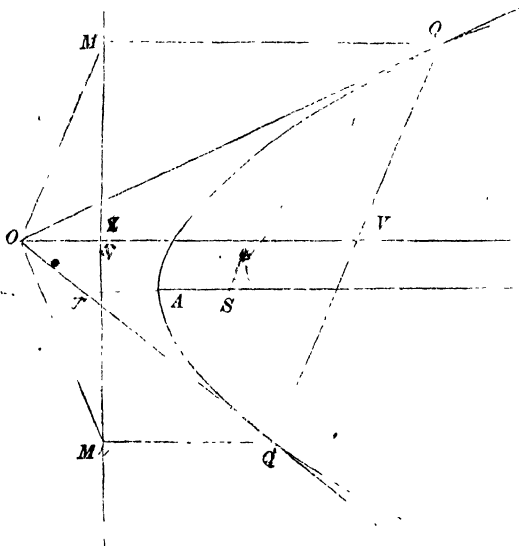
$$\therefore SQ \cdot SQ' = SO^2,$$

or  $SO$  is a mean proportional between  $SQ$  and  $SQ'$ .

$$OQ^2 : OQ'^2 :: SQ : SQ', \text{ and } QSOQ : Q'SOQ :: SQ : SQ',$$

### PROP. XIII.

15. If a pair of tangents  $OQ$ ,  $OQ'$  be drawn to a parabola, and  $OV$  be drawn parallel to the axis meeting  $QQ'$  in  $V$ , then  $QQ'$  shall be bisected in  $V$ .



Draw  $QM$ ,  $Q'M'$  at right angles to the directrix.

Join  $OM$ ,  $OM'$ ; and let  $OV$  meet  $MM'$  in  $Z$ .

Then, since  $OM = OM'$ , (*Prop. XI.*)

$\therefore$  the angle  $OMZ =$  the angle  $OM'Z$ ,

and the angle  $OZM =$  the angle  $OZM'$ ,

and the side  $OZ$  is common to the triangles  $OZM, OZM'$ ,

$$\therefore MZ = M'Z.$$

And because the lines  $QM, ZV, Q'M'$  are parallel,

$$\therefore QV : Q'V :: MZ : M'Z.$$

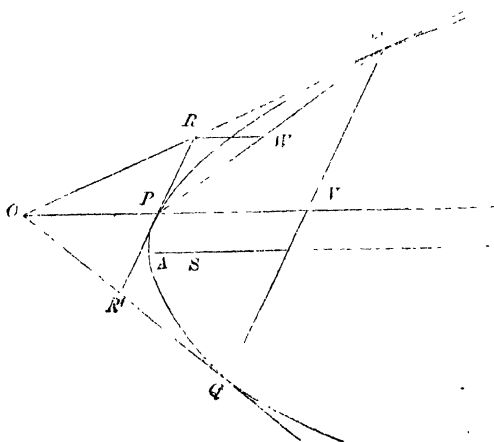
$$\text{But } MZ = M'Z,$$

$$\therefore QV = Q'V,$$

$$\therefore QQ' \text{ is bisected in } V.$$

#### PROP. XIV.

16. If from a point  $O$  a pair of tangents  $OQ, OQ'$  be drawn to a parabola, and  $OP$  be drawn parallel to the axis meeting the parabola in  $P$ , and  $QQ'$  in  $V$ , then the tangent at  $P$  will be parallel to  $QQ'$ , and  $OV$  will be bisected in  $P$ .



Draw the tangent  $RPR'$  meeting  $OQ, OQ'$  in  $R$  and  $R'$ .

Join  $PQ$ , and draw  $RW$  parallel to the axis, meeting  $PQ$  in  $W$ ;

Then, by the last Proposition,

$$PW = WQ.$$

And because  $RW$  is parallel to  $OP$ ,

$$\therefore OR : RQ :: PW : WQ.$$

$$\text{But } PW = WQ,$$

$$\therefore OR = RQ;$$

$$\text{so } OR' = R'Q',$$

$$\therefore OR : RQ :: OR' : R'Q',$$

$$\therefore RR' \text{ is parallel to } QQ'.$$

Again, since  $PR$  is parallel to  $QV$ ,

$$\therefore OP : PV :: OR : RQ.$$

$$\text{But } OR = RQ,$$

$$\therefore OP = PV.$$

COR. From this it is manifest that if any number of parallel chords be drawn in a parabola, their middle points will all lie on the line parallel to the axis which passes through the point where the tangent drawn parallel to the chords meets the parabola.

DEF. Any line  $PV$ , drawn from a point  $P$  in the parabola parallel to the axis, is called a *Diameter*.

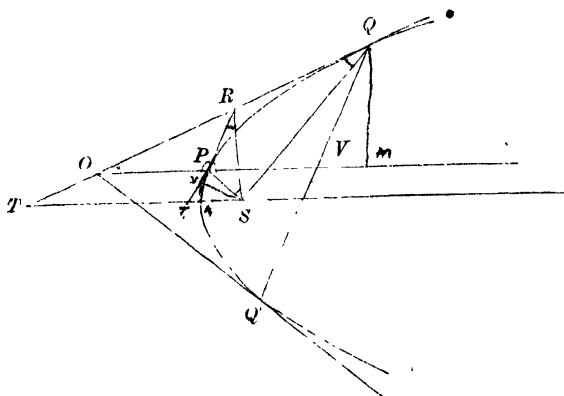
The point  $P$  is called the *Vertex* of the diameter  $PV$ ; and the tangent at  $P$  the *Tangent at the Vertex*.

The diameter consequently bisects all chords parallel to the tangent at the vertex, and the tangents at the extremities of any chord will intersect in the diameter corresponding to that chord.

DEF. A line  $QV$ , drawn parallel to the tangent at  $P$  from a point  $Q$  in the curve, is called the *Ordinate* to the diameter  $PV$ .

#### PROP. XV.

✓ 17. If  $QV$  be an ordinate to the diameter  $PV$ , then  $QV^2 = 4 \cdot SP \cdot PV$ .



Produce  $QV$  to meet the parabola in  $Q'$ ; and draw the tangents  $QO$ ,  $Q'O$ , meeting  $VP$  produced in the point  $O$ . (*Prop. XIV.*)

Also let the tangent at  $P$  meet  $OQ$  in  $R$ , and join  $SP$ ,  $SR$ , and  $SQ$ . Now since from the point  $R$  two tangents  $RP$ ,  $RQ$  are drawn to the parabola, the triangle  $RPS$  is similar to the triangle  $RSQ$ , (*Prop. XII.*)

$\therefore$  the angle  $SEP =$  the angle  $SQR$ .

But the angle  $SQR =$  the angle  $STQ$ , (*Prop. VII.*)  
 $=$  the angle  $POR$ ,

$\therefore$  the angle  $SRP =$  the angle  $POR$ ,

and the angle  $SPR =$  the angle  $OPR$ , (*Prop. V. Cor. 1*)

$\therefore$  the remaining angle  $RSP =$  the remaining angle  $QRP$ ,

$\therefore$  the triangle  $SPR$  is similar to the triangle  $POR$ ,

$$\therefore SP : PR :: PR : PO,$$

$$\therefore PR^2 = SP \cdot PO,$$

$$= SP \cdot PV. \quad (\text{Prop. XIV.})$$

Again, since  $QV$  is parallel to  $PR$ ,

$$\therefore QV : PR :: OV : OP.$$

$$\text{But } OV = 2 \cdot OP, \quad (\text{Prop. XIV.})$$



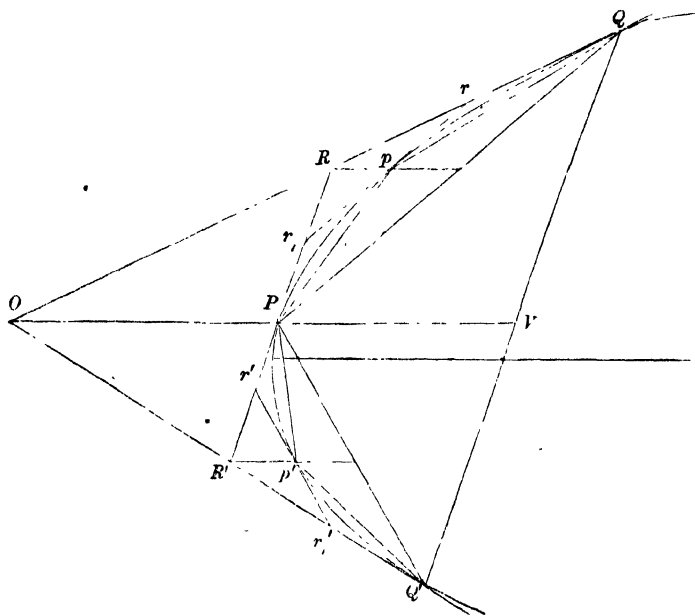




By *Euclid*, II. 6, the same may be proved to be true if the point  $O$  be without the parabola.

PROP. XVIII. (revisé)

20. If from an external point  $O$  a pair of tangents  $OQ$ ,  $OQ'$  be drawn to the parabola, and the chord  $QQ'$  be joined, the area of the figure bounded by  $QQ'$  and the curve is two-thirds of the triangle  $QQ'O$ .



Draw the diameter  $OV$  meeting the curve in  $P$ ; and let the tangent at  $P$  meet  $OQ$ ,  $OQ'$  in  $R$  and  $R'$ .

Join  $QP$ ,  $Q'P$ ; then

since  $OR = RQ$ ,

$\therefore$  the triangle  $OPR = \frac{1}{2}$  the triangle  $OPQ$ ,  
 $= \frac{1}{2}$  the triangle  $VPQ$ .

So the triangle  $OPR' = \frac{1}{2}$  the triangle  $VPQ$ ,

$\therefore$  the triangle  $ORR' = \frac{1}{2}$  the triangle  $PQQ'$ .

Again, if through  $R$  and  $R'$  we draw the diameters  $Rp$ ,  $R'p'$ ; and at the points  $p$  and  $p'$  draw the tangents  $rpr$ ,  $r'p'r'$ , we can prove in the same manner as before that

the triangle  $Rrr = \frac{1}{2}$  the triangle  $QpP$ ,

and the triangle  $R'r'r' = \frac{1}{2}$  the triangle  $Q'p'P$ .

Continuing in this manner to form new triangles by drawing diameters at the points  $r$ ,  $r$ , and  $r'$ ,  $r'$ , and tangents at the points where these diameters meet the curve, we can prove that the exterior triangles formed by the tangents are the halves of the interior triangles formed by joining the points of contact with the extremities of the chords.

And the same will hold however the number of the triangles be increased.

Hence the sum of all the exterior triangles will be equal to half the sum of all the interior triangles.

Now when the number of the triangles is increased indefinitely, the sum of the exterior triangles will represent the exterior figure  $OQPQ'$ , and the sum of the interior triangles the area of the interior figure  $QPQ'$ . Hence

the area of the figure  $OQPQ' = \frac{1}{2}$  the area of the figure  $QPQ'$ ,

$\therefore$  area of the figure  $OQPQ' = \frac{1}{3}$  the area of triangle  $QOQ'$ ,

$\therefore$  area of the figure  $QPQ' = \frac{2}{3}$  the area of triangle  $QOQ'$ .

21. DEF. If with a point  $O$  on the normal at  $P$  as centre and  $OP$  as radius, a circle be described touching the parabola at  $P$  and cutting it in  $Q$ ; then when the point  $Q$  is made to approach indefinitely near to  $P$ , the circle is called the *Circle of Curvature* at the point  $P$ . (See fig. Prop. XIX.)

## PROP. XIX.

The chord of the circle of curvature, at a point  $P$  of a parabola, drawn parallel to the axis  $= 4SP$ .

Let  $PT$  be the tangent, and  $PG$  the normal at the point  $P$ .

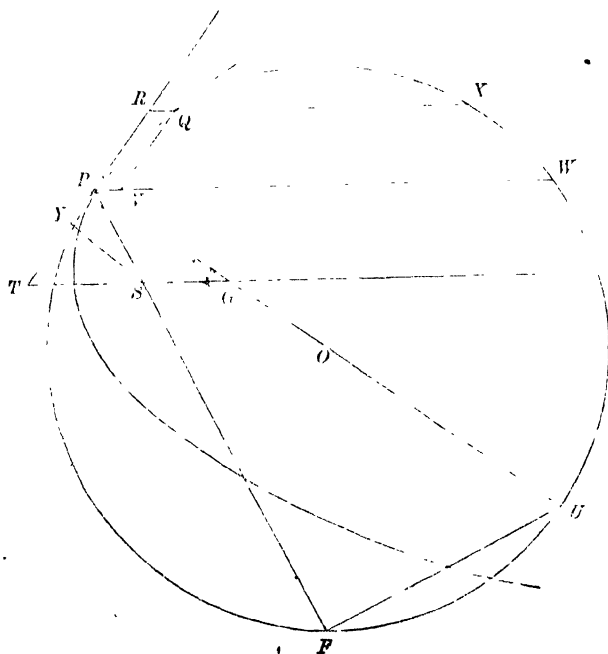
With centre  $O$  and radius  $OP$  describe a circle cutting the parabola in the point  $Q$ .

Draw  $RQX$  parallel to the axis meeting the circle in  $X$  and the tangent at  $P$  in  $R$ .

Also draw  $QV$  parallel to  $PR$ , and  $PW$  parallel to the axis; then

since  $RP$  touches the circle at  $P$ ,

$\therefore RP \cdot RX = PR^2$ . (*Euclid*, III. 36.)



But  $PR^2 = QV^2 = 4SP \cdot PV$ , (*Prop. XV.*)

$$\therefore RQ \cdot RX = 4SP \cdot PV.$$

But  $RQ = PV$ ,

$$\therefore RX = 4SP.$$

Now when the circle becomes the circle of curvature at  $P$ , the points  $R$  and  $Q$  move up to and coincide with  $P$ , and the lines  $RX$  and  $PW$  become equal.

Hence the chord of the circle of curvature parallel to the axis  $= 4SP$ .

COR. 1. If  $PU$  be the diameter of the circle of curvature, and  $PF$  the chord through the focus; then

since the angle  $FPU =$  the angle  $WPU$ , (*Prop. VIII.*)

$$\therefore PF = PW = 4SP.$$

COR. 2. If  $SY$  be drawn at right angles to  $PT$ ; then

the triangle  $PFU$  is similar to  $SYP$ ,

$$\therefore PU : PF :: SP : SY,$$

$$\text{or } PU : 4SP :: SP : SY.$$

### PROP. XX.

If  $QVQ'$  be any ordinate to the diameter  $PT$ , the circle described through the three points  $P$ ,  $Q$ ,  $Q'$  will intersect the parabola in a fourth point, which depends only upon the position of  $P$ .

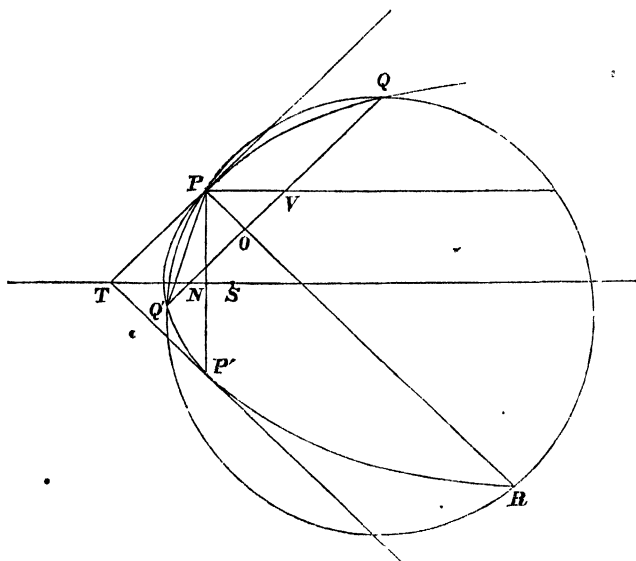
Draw the ordinate  $PN$ , and produce it to meet the parabola in  $P'$ ; then,

since the subtangent  $= 2 \cdot AN$ . (*Prop. VII.*)

The tangents at  $P$  and  $P'$  will meet the axis in the same point  $T$ .

Draw  $PR$  parallel to  $TP'$ , meeting the parabola in  $R$ , and  $QQ'$  in  $O$ ; then

$$PO \cdot OR : QO \cdot OQ' :: SP' : SP. \quad (\text{Prop. XVII.})$$



But  $SP = ST = SP'$ , (*Prop. VII.*)

$\therefore PO \cdot OR = QO \cdot OQ'$ .

Hence by the converse of *Euclid III. Prop. 22*, the point  $R$  is on the circle which passes through  $P, Q, Q'$ .

COR. 1. Since  $TP$  and  $TP'$  are equally inclined to the axis, the lines  $QQ', PR$ , which are parallel respectively to  $TP$  and  $TP'$ , are also equally inclined to the axis.

COR. 2. When the point  $V$  is brought indefinitely near to  $P$ ,  $QQ'$  coincides with the tangent to the parabola at  $P$ , and becomes also a tangent to the circle at  $P$ , since  $Q$  and  $Q'$  are indefinitely near to each other. The circle therefore becomes the circle of curvature at the point  $P$ .

Hence if  $PR$  be drawn parallel to the tangent at  $P'$ , or be equally inclined to the axis with  $PT$ , it will meet the parabola in the point where the circle of curvature at  $P$  intersects the parabola.

## PROBLEMS ON THE PARABOLA.

1. THE diameter of the circle described about the triangle  $BAC$  is equal to  $5AS$ . (See *fig. Prop. III.*)

2. If from the point  $G$ ,  $GK$  be drawn at right angles to  $SP$ , then  $PK = 2AS$ . (See *fig. Prop. VII.*)

3. If the triangle  $SPG$  is equilateral, then  $SP$  is equal to the latus rectum. (See *fig. Prop. VII.*)

4.  $PQ$  is a common tangent to a parabola and the circle described on the latus rectum as diameter; prove that  $SP$  and  $SQ$  make equal angles with the latus rectum.

5. Prove that  $PY \cdot PZ = SP^2$ , and that  $PY \cdot YZ = AS \cdot SP$ . (See *fig. Prop. VII.*)

6. If  $PL$  be drawn at right angles to  $AP$ , meeting the axis in  $L$ , and  $PN$  be the ordinate of  $P$ , then  $NL = 4AS$ .

7. The tangent at any point  $P$  of a parabola meets the directrix and latus rectum produced in points equally distant from the focus.

8. Prove that  $NY = TY$ , and that  $TP \cdot TY = TS \cdot TN$ . (See *fig. Prop. VII.*)

9. If a circle be described about the triangle  $SPN$ , the tangent to it from  $A = \frac{1}{2}PN$ . (See *fig. Prop. VII.*)

10. If the ordinate of a point  $P$  bisect the subnormal of  $P'$ , the ordinate of  $P$  is equal to the normal of  $P'$ .

11. If from any point on the tangent to a parabola a line be drawn touching the parabola, the angle between this line and the line to the focus from the same point is constant.

12. A circle and parabola have the same vertex and axis.  $BA'C$  is the double ordinate of the parabola which touches the circle at  $A'$ , the extremity of the diameter through the vertex  $A$ .  $PP'$  is any other ordinate of the parabola parallel to this, meeting the axis in  $N$ , and  $AB$  produced in  $R$ ; prove that the rectangle  $RP \cdot RP'$  is proportional to the square of the tangent drawn from  $N$  to the circle.

13. Draw a parabola to touch a given circle at a given point, and such that its axis may touch the same circle in another given point.

14. If from the point of contact of a tangent to a parabola a chord be drawn, and another line be drawn parallel to the axis meeting the chord, tangent, and curve, this line will be divided by them in the same ratio as it divides the chord.

15. If the diameter  $PV$  meet the directrix in  $O$ , and the chord drawn through the focus parallel to the tangent at  $P$  in  $V$ ; prove that  $PV = PO$ .

16. Prove that the locus of the intersection of a diameter  $PV$  with the chord drawn through the focus parallel to the tangent at  $P$  is a parabola.

17. If a circle and parabola have a common tangent at  $P$ , and intersect in  $Q$  and  $R$ ; and  $QV$ ,  $UR$  be drawn parallel to the axis of the parabola meeting the circle in  $V$  and  $U$  respectively, then  $VU$  is parallel to the tangent at  $P$ .

18.  $AB$  and  $AC$  are two lines at right angles to each other. From a fixed point  $C$  on  $AC$ ,  $CR$  is drawn parallel to  $AB$ . On  $AR$ , produced if necessary,  $P$  is taken such that the perpendicular  $PN$  upon  $AB$  is equal to  $CR$ . Prove that the curve traced out by  $P$  is a parabola.

19. If from a point  $P$  of a circle  $PC$  be drawn to the centre; and  $R$  be the middle point of the chord  $PQ$  drawn parallel to a fixed diameter  $ACB$ , then the curve traced out by the intersection of  $CP$  and  $AR$  is a parabola.

20. If two equal tangents  $OQ$ ,  $OQ'$  be cut by a third tangent, their alternate segments are equal.

21.  $E$  is the centre of the circle described about the triangle  $OQ'Q$ . Prove that the circle described about the triangle  $QE'Q$  will pass through the focus. (See fig. Prop. XIII.)

22.  $PSp$  is any focal chord of a parabola. Prove that  $AP$ ,  $Ap$  will meet the latus rectum in two points  $Q, q$ , whose distances from the focus are equal to the ordinates of  $p$  and  $P$ .

23.  $PSp$  is a focal chord of a parabola,  $RDr$  the directrix meeting the axis in  $D$ ; and  $Q$  any point on the curve. Prove that if  $QP, Qp$  be produced to meet the directrix in  $R, r$ , half the latus rectum is a mean proportional between  $DR, Dr$ .

24.  $OP$  and  $OQ$  are two tangents to a parabola. On  $QO$  produced,  $OQ'$  is taken equal to  $OQ$ ; prove that  $OS \cdot PQ' = OP \cdot OQ$ .

25. If  $QD$  be drawn at right angles to the diameter  $PV$ , then  $QD^2 = 4AS \cdot PV$ .

26. If through any point  $O$  on the axis of a parabola a chord  $POQ$  be drawn, and  $PM, QN$  be the ordinates of the points  $P$  and  $Q$ , prove that  $AM \cdot AN = AO^2$ .

27. If  $AP$  and  $AQ$  be drawn at right angles to each other from the vertex of a parabola, and  $PM, QN$  be the ordinates of  $P$  and  $Q$ , prove that the latus rectum is a mean proportional between  $AM$  and  $AN$ .

28.  $OAP$  is the sector of a circle whose centre is  $O$ . If the radius  $OA$  remain fixed while the angle  $AOP$  changes, the centre of the circle inscribed in the sector,  $AOP$  will trace out a parabola.

29.  $QS'Q'$  is a focal chord parallel to  $AP$ ;  $PN, QM, Q'M'$  are the ordinates of  $P, Q$ , and  $Q'$ . Prove that  $SM^2 = AM \cdot AN$  and that  $MM' = AP$ .

30.  $PQ, PQ'$  are drawn from any point  $P$  cutting the ordinates  $Q'V', QV$  in  $R'$  and  $R$ , prove that  $VR$  is to  $V'R'$  in the triplicate ratio of  $QV$  to  $Q'V'$ .



31. On a chord of a parabola as diameter a circle is described cutting the parabola again in two points. If these points be joined, the portion of the axis between the two chords is equal to the latus rectum.

32. If  $OQ$ ,  $OQ'$  be a pair of tangents to a parabola, and the chord  $QQ'$  be a normal to the curve at  $Q$ , then  $OQ$  is bisected by the directrix.

33. Two equal parabolas having the same focus and their axes in contrary directions intersect at right angles.

34. The radius of curvature at the extremity of the latus rectum is equal to twice the normal.

35. If from any point  $P$  of a parabola  $PF$  and  $PH$  be drawn making equal angles with the normal  $PG$ , then  $SG^2 = SF \cdot SH$ .

36. If a triangle be inscribed in a parabola, the points when the sides produced meet the tangents at the opposite angles are in the same straight line.

37. If the tangents  $OQ$ ,  $OQ'$  be cut by a third tangent in  $R$ ,  $R'$ , prove that

$$OR : RQ :: R'Q' : OR'.$$

38. If from the vertex of a parabola chords be drawn at right angles to one another, and on them a rectangle be described, the curve traced out by the further angle is a parabola.

39. Prove that  $2PY$  is a mean proportional between  $AP$  and the chord of the circle of curvature at the point  $P$  of the parabola drawn through the vertex  $A$ . (*See fig. Prop. VII.*)

40. If a circle described upon the chord of a parabola as diameter meet the directrix it also touches it, and all chords for which this is possible intersect in a point.

41. If a parabola *roll* upon another equal parabola, the vertices originally coinciding, the focus traces out the directrix.

42. The circle of curvature at the extremity of the latus rectum intersects the parabola on the diameter of curvature passing through the point of contact.

## CHAPTER II.

### THE ELLIPSE.

22. DEF. The *Ellipse* is the curve traced out by a point which moves in such a manner that its distance from a given fixed point continually bears the same ratio, *less than unity*, to its distance from a given fixed line. (*See Introduction.*)

#### PROP. I.

The focus and directrix of an ellipse being given, to find any number of points on the curve.

Let  $S$  be the focus and  $MX$  the directrix.

Draw  $SX$  at right angles to the directrix, and divide  $SX$  in the point  $A$ , so that  $SA$  may be to  $AX$  in the given fixed ratio less than unity; then

$A$  is a point on the curve.

On  $XS$  produced take a point  $A'$ , such that

$$SA' : A'X :: SA : AX;$$

then  $A'$  will also be a point on the curve.

On the directrix take *any* point  $M$ ; and through  $M$  and  $S$  draw the line  $MYSY'$ , meeting  $AY$  and  $A'Y'$ , drawn at right angles to  $AA'$ , in the points  $Y$  and  $Y'$ .

On  $YY'$  as diameter describe a circle, and draw  $MPP'$  parallel to  $AA'$ , cutting the circle in the points  $P$  and  $P'$ ;

$P$  and  $P'$  will be points on the ellipse.



In the same way, by taking other points on the directrix, we may obtain as many more points on the curve as we please.

COR. 1. Since, corresponding to every point  $P$  on the curve, there is a point  $P'$  situated in precisely the same manner with respect to  $A'Y'$  as  $P$  is with respect to  $AY$ , it is clear that if we make  $A'S'$  equal to  $AS$ , and  $A'X'$  equal to  $AX$ , and draw  $X'M'$  at right angles to  $A'X'$ , the curve could be equally well described with  $S'$  as focus and  $M'X'$  as directrix.

The ellipse is therefore symmetrical, not only with respect to the line  $AA'$ , but also with respect to the line  $OC$  drawn through the middle point of  $YY'$  at right angles to and bisecting  $AA'$ .

COR. 2. The line  $OP$  will bisect the angle  $SPS'$ .

Let  $OP$  meet  $SS'$  in  $G$ . Produce  $MP$  to meet  $X'M'$  in  $M'$ , and draw  $OM'$  passing through the focus  $S'$ ; then

$$SP : PM :: S'P : PM',$$

or, alternately,  $SP : S'P :: PM : PM'$ . (1)

$$\text{Again, } SG : PM :: S'G : PM',$$

or, alternately,  $SG : S'G :: PM : PM'$ , (2)

$\therefore$  from (1) and (2)

$$SP : S'P :: SG : S'G,$$

$\therefore PG$  bisects the angle  $SPS'$ . (*Euclid*, VI. 3.)

It will be shown hereafter (*Prop. XI.*) that the normal to the ellipse at the point  $P$  also bisects the angle  $SPS'$ . Hence the ellipse and circle have the same tangent at the point  $P$ .

The ellipse will consequently touch all the infinite series of circles which can be described in the same manner as the one in the figure by taking different points on the directrix.

PROP. II.

23. If  $C$  be the middle point of  $AA'$ , then  $CA$  is a mean proportional between  $CS$  and  $CX$ ,

or  $CS \cdot CX = CA^2$ . (See fig. Prop. III.)

Since  $SA' : A'X :: SA : AX$ .

Alternately  $SA' : SA :: A'X : AX$ ,

$\therefore SA' + SA : SA :: A'X + AX : AX$ ;

or  $AA' : SA :: XX' : AX$ ,

$\therefore AA' : XX' :: SA : AX$ .

or  $CA : CX :: SA : AX$ . (1)\*

Again,  $SA' : SA :: A'X : AX$ ,

$\therefore SA' - SA : SA :: A'X - AX : AX$ ;

or  $SS' : SA :: AA' : AX$ .

Alternately  $SS' : AA' :: SA : AX$ ;

or  $CS : CA :: SA : AX$ . (2)

Hence from (1) and (2)

$CA : CX :: CS : CA$ ,

$\therefore CA^2 = CX \cdot CS$ ;

or  $CA$  is a mean proportional between  $CS$  and  $SX$ .

COR. Since the three lines  $CS$ ,  $CA$ ,  $CX$  are proportional, therefore, by the definition of duplicate ratio and *Euclid*, VI. 20 Cor.

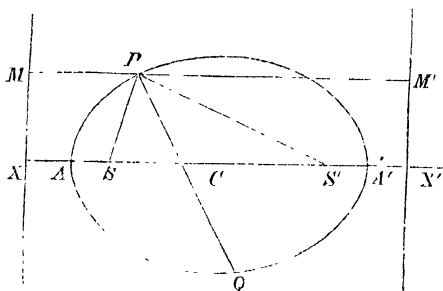
$CA : CX :: CS^2 : CA^2$ . (3)

PROP. III.

24. If  $P$  be any point on the ellipse, then

$$SP + S'P = AA'.$$

\* N.B. The results (1), (2), (3), should be remembered, as they will frequently be referred to.



Since  $SP : PM :: SA : AX$ ,  
 and  $AX : AX :: AA' : XX'$ , (*Prop. II.*)  
 $\therefore SP : PM :: AA' : XX'$ ,  
 So  $S'P : PM' :: AA' : XX'$ ,  
 $\therefore SP + S'P : PM + PM' :: AA' : XX'$ .  
 But  $PM + PM' = MM' = XX'$ ,  
 $\therefore SP + S'P = AA'$ .

*Cor. 1.* By means of this property the ellipse may be practically described and the form of the curve determined.

Let a string, equal in length to  $AA'$ , have its ends fastened to two points  $S$  and  $S'$ ; and let it be kept stretched by means of the point of a pencil at  $P$ ; then since  $SP + S'P$  will be always equal to  $AA'$ , the point  $P$  will trace out the ellipse.

*Cor. 2.* The line  $AA'$  is the longest line that can be drawn in the ellipse.

For, if any other line  $PQ$  be drawn, then

$$\begin{aligned} SP + S'Q &> PQ, \\ \text{and } S'P + S'Q &> PQ, \\ \therefore SP + S'P + S'Q + S'Q &> 2PQ, \\ \text{or } AA' &> PQ. \end{aligned}$$

25. *DEF.* If  $BCB'$  be drawn at right angles to  $ACA'$ , meeting the ellipse in  $B$  and  $B'$ , it will be seen further on (*Prop. XIII. Cor. 2*) that  $BCB'$  is the shortest chord that can be drawn through the *centre* of the ellipse. (*See fig. Prop. IV.*)

$AA'$  is called the *Major Axis* and  $BB'$  the *Minor Axis* of the ellipse.

In most geometrical treatises the ellipse is defined as the curve traced out by a point which moves in such a manner that the sum of its distances from two fixed points is always the same; but it appears that the properties of the curve are more clearly exhibited by defining it in a manner analogous to the parabola, and deducing *immediately* from that definition the property in question.

Having now shown that one definition necessarily includes the other, we are at liberty in our future investigations to make use of whichever property is most convenient.



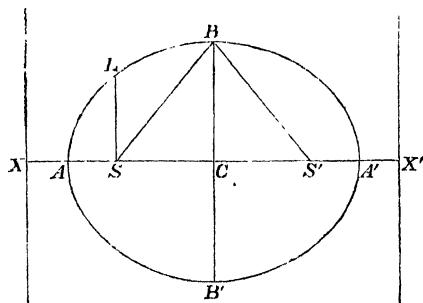
#### PROP. IV.

26. If  $BC$  be the semi-minor axis of the ellipse, then

$$BC^2 = CA^2 - CS^2;$$

and if  $SL$  be the semi-latus rectum,

$$SL \cdot AC = BC^2.$$



Join  $SB, S'B$ ; then

since  $SB + S'B = AA'$ , (*Prop. III.*)

and that  $SB = S'B$ ,

$$\therefore SB = AC.$$

$$\text{But } BC^2 = SB^2 - CS^2,$$

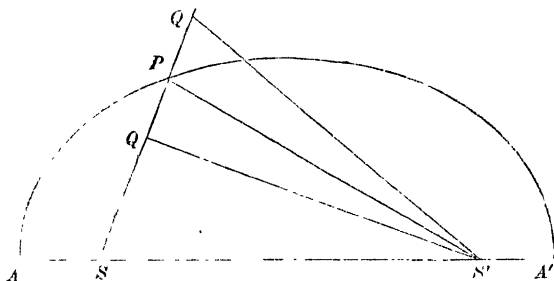
$$\therefore BC^2 = CA^2 - CS^2.$$

Again,  $SL : SX :: SA : AX,$   
 $:: CS : CA, (Prop. II.)$

$$\begin{aligned} \therefore SL \cdot AC &= CS \cdot SX, \\ &= CS \cdot CX - CS^2, \text{ (Euclid, II. 3)} \\ &= CA^2 - CS^2, \text{ (Prop. II.)} \\ &= BC^2. \end{aligned}$$

PROP. V.

27. The sum of the distances of *any* point from the foci of an ellipse will be less or greater than  $AA'$  according as the point is inside or outside the ellipse.



(1.) Let  $Q$  be a point inside the ellipse.

Join  $SQ, S'Q$ ; and produce  $SQ$  to meet the ellipse in  $P$ ; join  $S'P$ ; then

since  $S'P + QP > S'Q$ ,

$$\therefore S'P + SP > S'Q + SQ.$$

But  $S'P + SP = AA'$ , (*Prop.* III.)

$$\therefore SQ + S'Q < AA'.$$

(2.) Let  $Q$  be a point outside the ellipse.

Join  $SQ$ ,  $S'Q$ , and let  $SQ$  meet the ellipse in the point  $P$ ; join  $S'P$ ; then

since  $S'Q + QP > S'P$ ,

$$\therefore S'Q + SQ > SP + S'P,$$

But  $SP + S'P = AA'$ , (*Prop.* III.)

$$\therefore SQ + S'Q > AA'.$$





Produce  $PS$  to meet the ellipse in the point  $Q$ ; then since the triangles  $ZMP$ ,  $ZM'P'$  are similar,

$$\therefore ZP : ZP' :: MP : M'I',$$

$$:: SP : S'P',$$

$\therefore SZ$  bisects the angle  $P'SQ$ . (*Euclid*, VI. Prop. A.)

Now when  $P'$  is indefinitely near to  $P$ , and  $PI'$  becomes the tangent at the point  $P$ , the angle  $PSP'$  becomes indefinitely small, while the angle  $QSI'$  approaches two right angles; and therefore the angle  $ZSP'$ , being half of the angle  $P'SQ$ , becomes ultimately a right angle.

Hence when  $PZ$  becomes the tangent at the point  $P$ ,

the angle  $ZSP$  is a right angle,

or  $SZ$  is perpendicular to  $SP$ .

COR. 1. Conversely, if  $SZ$  be drawn at right angles to  $SP$  meeting the directrix in  $Z$ , and  $PZ$  be joined,  $PZ$  will be the tangent at  $P$ .

COR. 2. If  $ZP$  be produced to meet the other directrix in the point  $Z'$ , and  $S'Z'$  be joined, then

$S'Z'$  is at right angles to  $S'P$ .

COR. 3. The tangents at the extremities of the latus rectum or double ordinate through the focus meet the axis produced in the point  $X$ .

#### PROP. VII.

The tangent to the ellipse at any point  $P$  makes equal angles with the focal distances  $SP$  and  $S'P$ .

Let the tangent at  $P$  meet the directrices in  $Z$  and  $Z'$ .

Draw  $MPM'$  at right angles to the directrices, meeting them in  $M$  and  $M'$  respectively; join  $SZ$ ,  $S'Z'$ ; then

$$SP : PM :: S'P : PM';$$

and since the triangles  $MPZ$ ,  $M'PZ'$ , are similar,

$$PM : PZ :: PM' : PZ',$$

$$\therefore SP : PZ :: S'P : PZ'. \quad (\text{Ex æquali.})$$

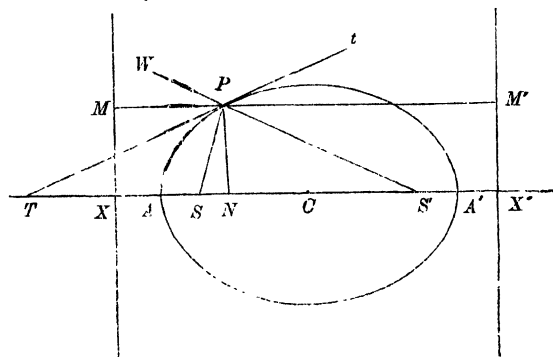


the angle  $ZSP$  is a right angle, (*Prop. VI.*)  
 and  $\therefore$  also the angle  $ZSQ$  is a right angle,  
 $\therefore ZQ$  is the tangent at  $Q$ ; (*Prop. VI. Cor. 1*)  
 or the tangents at the extremities of a focal chord intersect in the directrix.

## PROP. IX.

29. If the tangent at  $P$  meet the axis major produced in  $T$ , and  $PN$  be the ordinate of the point  $P$ , then •

$$CT \cdot CN = CA^2.$$



Draw  $MPM'$  parallel to the axis major meeting the directrices in  $M$  and  $M'$ ; and produce  $S'P$  to  $W$ ; then, since  $P'T$  bisects the angle  $SPW$ , (*Prop. VII. Cor.*)

$$\therefore S'T : ST :: S'P : SP, \quad (\text{Euclid, VI. A.})$$

$$:: PM' : PM,$$

$$:: X'N : XN,$$

$$\therefore S'T + ST : S'T - ST :: X'N + XN : X'N - XN;$$

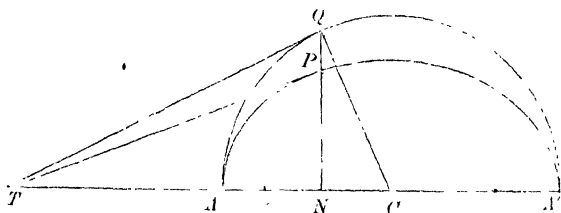
$$\text{or } 2CT : 2CS :: 2CX : 2CN,$$

$$\text{or } CT : CS :: CX : CN,$$

$$\therefore CT \cdot CN = CS \cdot CX, \\ = CA^2. \quad (\text{Prop. II.})$$

## PROP. X.

If on the major axis of an ellipse as diameter a circle be described and a common ordinate  $NPQ$  be drawn meeting the ellipse in  $P$  and the circle in  $Q$ , then the tangents to the ellipse and circle respectively at the points  $P$  and  $Q$  will meet the major axis produced in the same point.)



Let the tangent to the ellipse at  $P$  meet the major axis produced in  $T$ ; join  $CQ$ ,  $QT$ ; then, by the last Proposition,

$$CT \cdot CN = CA^2 = CQ^2,$$

$\therefore$  the angle  $CQT$  is a right angle.

And therefore  $QT$  is the tangent to the circle at  $Q$ ; ~~or~~ the tangents at  $P$  and  $Q$  meet the major axis produced in the same point  $T$ .

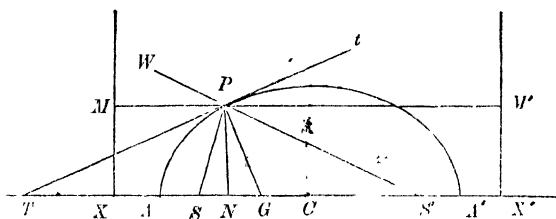
The circle described on  $AA'$  as diameter is called the *Auxiliary Circle* on account of the important aid that it affords in investigating the properties of the ellipse.

30. DEF. The line  $PG$ , drawn at right angles to the tangent  $PT$ , is called the *Normal* to the ellipse at the point  $P$ .

## PROP. XI.

If the normal at  $P$  meet the major axis in the point  $G$ ; then

$$SG : SP :: CS : CA.)$$



Since  $PG$  is at right angles to  $TT'$ ,

$\therefore$  the angle  $GPT =$  the angle  $GPT'$ .

But the angle  $SPT =$  the angle  $S'Pt$ , (*Prop. VII.*)

$\therefore$  the angle  $SPG =$  the angle  $S'PG$ ,

or  $PG$  bisects the angle  $SPS'$ ,

$\therefore SG : S'G :: SP : S'P$ , (*Euclid, VI. 3.*)

$\therefore SG : SG + S'G :: SP : SP + S'P$ ;

or  $SG : SS' :: SP : AA'$ ;

or  $SG : SP :: SS' : AA'$ ;

or  $SG : SP :: CS : CA$ .

Cor. Hence also,

$$S'G : S'P :: CS : CA.$$

### PROP. XII.

31. If the normal at  $P$  meet the major axis in  $G$ , and  $PN$  be the ordinate at the point  $P$ , then (*see fig. Prop. XI.*)

$$NG : NC :: BC^2 : AC^2.)$$

Draw  $MPM'$  parallel to the axis meeting the directrices in  $M$  and  $M'$ ; join  $SP, S'P$ ; then, since  $PG$  bisects the angle  $SPS'$ , (*Prop. XI.*)

$$\therefore S'G : SG :: S'P : SP,$$

$$:: PM' : PM,$$

$$:: X'N : XN,$$

$$\therefore S'G - SG : S'G + SG :: X'N - XN : X'N + XN;$$

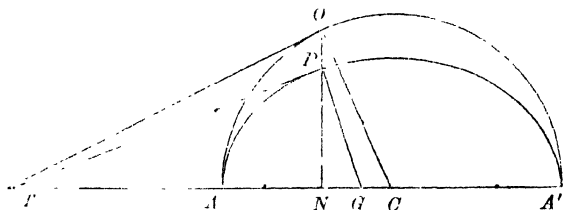
$$\text{or } 2CG : SS' :: \frac{1}{2}CN : XX'.$$

Alternately,  $2\,CG : 2\,CN :: SS' : XX'$ ;  
 or  $CG : CN :: CS : CX$ ,  
 $:: CS^2 : CA^2$ , (*Prop. II. Cor.*)  
 $\therefore CN - CG : CN :: CA^2 - CS^2 : CA^2$ ;  
 or  $NG : CN :: BC^2 : AC^2$ .

## PROP. XIII.

✓ 32. If  $PN$  be the ordinate of any point  $P$  on the ellipse ;  
 then

$$PN^2 : AN \cdot A'N :: BC^2 : AC^2.)$$



Produce  $NP$  to meet the auxiliary circle in the point  $Q$ , and draw the tangents  $PT$ ,  $QT$  meeting the major axis produced in the point  $T$ . (*Prop. X.*)

Join  $CQ$ , and let the normal at  $P$  meet the ellipse in  $G$  ; then, by the last Proposition,

$$NG : CN :: BC^2 : AC^2.$$

And rectangles of the same altitude are to another as their bases,

$$\therefore TN \cdot NG : TN \cdot CN :: BC^2 : AC^2 ;$$

$$\text{or } PN^2 : QN^2 :: BC^2 : AC^2. \quad (\text{Euclid, VI. 8, Cor.})$$

$$\text{But } QN^2 = AN \cdot A'N,$$

since the angle  $AQA'$  in a semicircle is a right angle,

$$\therefore PN^2 : AN \cdot A'N :: BC^2 : AC^2.$$

COR. 1. Also

$$PN : QN :: BC : AC.$$

This result is the basis of many of the future Propositions of the ellipse.

COR. 2. Since  $PN^2 : QN^2 :: BC^2 : AC^2$ ,  
 $\therefore PN^2 : AC^2 - CN^2 :: BC^2 : AC^2$ ,  
 $\therefore PN^2 : AC^2 - CN^2 - PN^2 :: BC^2 : AC^2 - BC^2$ ,  
 or  $PN^2 : AC^2 - CP^2 :: BC^2 : AC^2 - BC^2$ .

Now  $PN^2$  is always less than  $BC^2$ ,

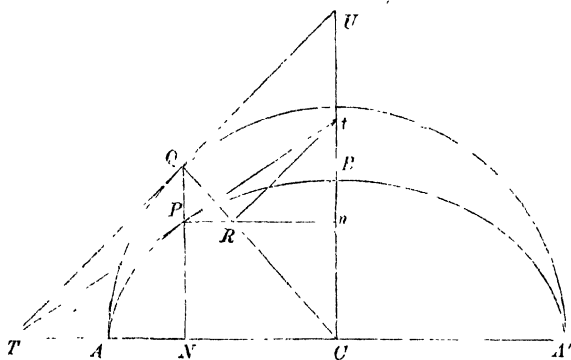
$\therefore CP^2$  is always greater than  $BC^2$ ,

$\therefore BC$  is the shortest line that can be drawn to the ellipse from the centre.

#### PROP. XIV.

If the tangent at any point  $P$  of an ellipse meet the minor axis  $CB$  produced in  $t$ , and  $Pn$  be drawn at right angles to  $CB$ ; then

$$Ct \cdot Cn = BC^2$$



Draw the common ordinate  $NPQ$  to the ellipse and the auxiliary circle; and let the tangents at  $P$  and  $Q$  to the ellipse and circle respectively meet the major axis produced in  $T$  (*Prop. X.*) and the minor axis produced in  $t$  and  $U$ .

Join  $CQ$  meeting  $Pn$  in  $R$ ; then since  $PR$  is parallel to  $CN$ ,

$$\begin{aligned} CR : CQ &:: PN : QN, \\ &:: BC : AC. \quad (\text{Prop. XIII. Cor. 1.}) \end{aligned}$$



But  $CQ = AC$ ,

$$\therefore CR = BC. \quad (1)$$

Again, joining  $Rt$ ,

$$Ct : CU :: PN : QN, \quad —$$

$$:: CR : CQ,$$

$\therefore Rt$  is parallel to  $QU$ ,

$\therefore CRt$  is a right angle,

$$\therefore Ct \cdot Cn = CR^2. \quad (\text{Euclid, VI. 8, Cor})$$

$$\text{But } CR = BC, \quad (1)$$

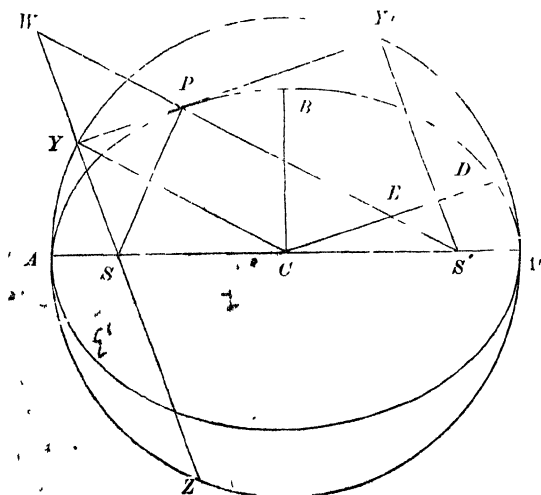
$$\therefore Ct \cdot Cn = BC^2.$$

This proposition also admits of a demonstration similar to that given for the corresponding property of the hyperbola.

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### PROP. XV.

(33. If from the foci  $S$  and  $S'$ ,  $SY$  and  $S'Y'$  are drawn at right angles to the tangent at  $P$ , then  $Y$  and  $Y'$  are on the circumference of the auxiliary circle, and  
 $SY \cdot S'Y' = BC^2$ .)



Join  $SP$ ,  $S'P$ , and produce  $SY$  and  $S'P$  to meet in  $W$ ; and join  $CY$ ; then

since the angle  $SPY =$  the angle  $WPY$ , (*Prop. VII. Cor.*)

and the angle  $SY P =$  the angle  $WYP$ ,

and the side  $PY$  is common to the triangles  $SPY$ ,  $WPY$ ,

$\therefore$  the triangle  $SPY =$  the triangle  $WPY$  in all respects,

$$\therefore SP = PW,$$

$$\therefore SP + S'P = S'W.$$

$$\text{But } SP + S'P = AA', \quad (\text{Prop. III.})$$

$$\therefore S'W = AA'.$$

Again  $\therefore SC = S'C$  and  $SY = YW$ ,

$$\therefore SC : S'C :: SY : YW,$$

$\therefore CY$  is parallel to  $S'W$ ,

$$\therefore CY : S'W :: CS : SS',$$

$$\therefore CY = \frac{1}{2} S'W = CA.$$

$$\text{So } CY = CA,$$

$\therefore Y$  and  $Y'$  are points on the auxiliary circle.

Next let  $YS$  be produced to meet the auxiliary circle in  $Z$ , and join  $ZY'$ ; then

since the angle  $ZYY'$  is a right angle,

$\therefore ZY'$  passes through the centre  $C$ ,

$\therefore$  the angle  $SCZ =$  the angle  $S'CY'$ .

$$\therefore SZ = S'Y',$$

$$\therefore SY \cdot S'Y' = SY \cdot SZ,$$

$$= AS \cdot A'S, \quad (\text{Euclid, III. 35})$$

$$= CA^2 - CS^2, \quad (\text{Euclid, II. 5})$$

$$= BC^2. \quad (\text{Prop. IV.})$$

**COR.** If  $CD$  be drawn parallel to the tangent at  $F$ , meeting  $S'P$  in  $E$ ; then

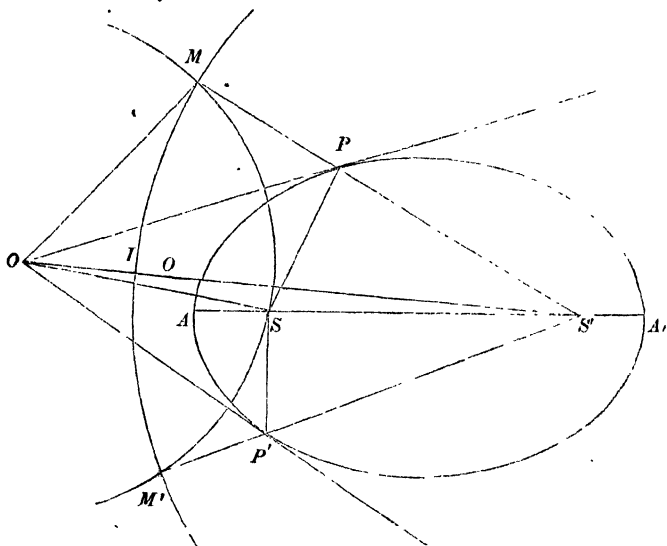
since the figure  $CYPE$  is a parallelogram,

$$\therefore PE = CY = AC.$$



# PROP. XVI.

34. To draw a pair of tangents to an ellipse from an external point  $O$ .



With centre  $S'$  and radius equal to  $AA'$  describe a circle.

Join  $OS$ ,  $OS'$ ; and let  $S'O$  or  $S'O$  produced meet the circle in the point  $I$ .

Now, if  $O$  be a point outside the circle  $MIM'$ , it is evident that  $OS$  is greater than  $OI$ ; and if  $O$  be inside the circle,

$$\text{since } OS + OS' > AA' \text{ or } S'I, \quad (\text{Prop. V.})$$

$$\therefore OS > OI.$$

With centre  $O$  and radius  $OS$  describe another circle cutting the former in the points  $M$  and  $M'$ , which it will always do since  $OS$  is greater than  $OI$ .

Join  $S'M$ ,  $S'M'$ , meeting the ellipse in the points  $P$  and  $P'$ .

Join  $OP$ ,  $OP'$ ; these will be the tangents required.

Join  $SP$ ,  $SP'$ ; then since

$$SP + S'P = AA' = S'M,$$

$$\therefore SP = PM.$$

And  $\therefore SP$ ,  $PO = MP$ ,  $PO$  each to each,

$$\text{and } OS = OM,$$

$\therefore$  the angle  $OPS =$  the angle  $OPM$ ,

$\therefore OP$  is the tangent at  $P$ . (*Prop. VII. Cor.*)

So  $OP'$  is the tangent at  $P'$ . •

#### PROP. XVII.

(If from a point  $O$  a pair of tangents  $OP$ ,  $OP'$  be drawn to an ellipse, then  $OP$  and  $OP'$  will subtend equal angles at either focus.)

Join  $SP$ ,  $S'P$ ;  $SP'$ ,  $S'P'$ ; and produce  $S'P$ ,  $S'P'$  to  $M$  and  $M'$ , making  $PM$  equal to  $SP$ , and  $P'M'$  equal to  $SP'$ .

Join  $OM$ ,  $OM'$ ;  $OS$ ,  $OS'$ .

Then since  $OP$ ,  $PS = OP$ ,  $PM$ , each to each,

and the angle  $OPS =$  the angle  $OPM$ , (*Prop. VII. Cor.*)

$$\therefore OS = OM,$$

and the angle  $OSP =$  the angle  $OMP$ .

$$\text{So } OS = OM',$$

and the angle  $OSP' =$  the angle  $OM'P'$ ,

$$\therefore OM = OM'.$$

Again,  $\therefore S'M = S'P + SP = AA'$ ,

and  $S'M' = S'P' + SP' = AA'$ .

$$\therefore S'M = S'M'.$$

And  $\therefore OS'$ ,  $S'M = OS'$ ,  $S'M'$ , each to each,

$$\text{and } OM = OM',$$

$\therefore$  the angle  $OS'M =$  the angle  $OS'M'$ ,

and the angle  $OMS' =$  the angle  $OM'S'$ .

But the angle  $OMS' =$  the angle  $OSP$ ,  
 and the angle  $OM'S' =$  the angle  $OSP'$ ,  
 $\therefore$  the angle  $OSP =$  the angle  $OSP'$ ,  
 $\therefore OP$  and  $OP'$  subtend equal angles at either focus.

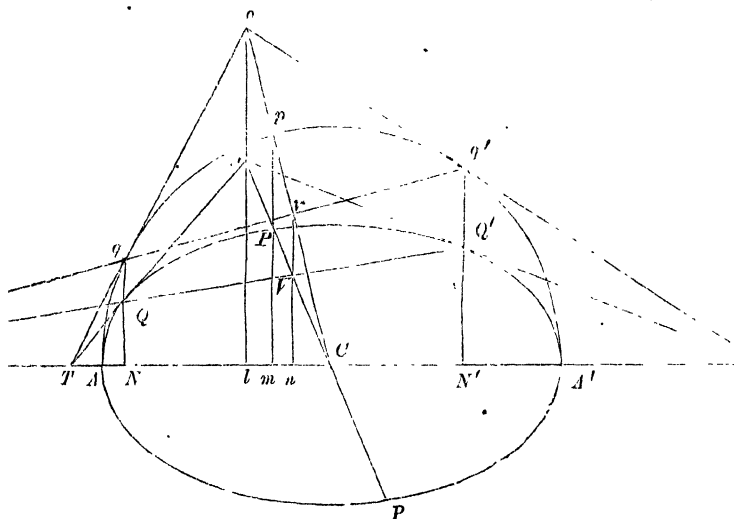
○ PROP. XVIII. *n. Boole.*

{ 35. If from an external point  $O$  a pair of tangents  $OQ$ ,  $OQ'$  be drawn to an ellipse, and  $CO$  be joined meeting the chord  $QQ'$  in  $V$ , and the ellipse in  $P$ ; then

(1.)  $QQ'$  will be bisected in  $V$ .

(2.) The tangent at  $P$  will be parallel to  $QQ'$ .

(3.)  $CP$  will be a mean proportional between  $CO$  and  $CO'$ .)



Produce  $OQ$ ,  $OQ'$  to meet the major axis produced in  $T$  and  $T'$ .

Draw the ordinate  $NQ$ ,  $N'Q'$ , and produce them to meet the circle in  $q$  and  $q'$ .

Then  $Tq$  and  $T'q'$  will be tangents to the auxiliary circle.  
(*Prop. X.*)

Let  $Tq$  and  $T'q'$  be produced to meet in  $o$ ; join  $Co$  meeting the chord  $qq'$  in  $v$ , and the circle in  $p$ .

Now, since the corresponding ordinates of the ellipse and auxiliary circle are in the constant ratio of  $BC$  to  $AC$ , the three lines  $ol$ ,  $pm$ ,  $vn$  drawn at right angles to  $AA'$  will pass through the points  $O$ ,  $P$ ,  $V$  respectively.

For, according as  $O$  is the point where  $ol$  meets  $TQ$  or  $T'Q'$  we shall have

$$\begin{aligned} lO : lo &:: NQ : Nq, \\ &:: BC : AC; \\ \text{or } lO : lo &:: N'Q' : N'q', \\ &:: BC : AC, \end{aligned}$$

$\therefore Oo$  is perpendicular to  $AA'$ .

So  $Pp$  and  $Vv$  are perpendicular to  $AA'$ ,

$\therefore Oo, Pp, Vv$  are parallel.

Hence (1.)  $QV : VQ' :: qv : vq'$ .

But  $qv = vq'$  from the circle,

$\therefore QV = VQ'$ ;

or  $QQ'$  is bisected in  $V$ .

(2.) Since  $NQ : Nq :: N'Q' : N'q'$

it is evident that  $QQ'$  and  $qq'$  will meet the axis produced in the same point.

Also the tangents to the ellipse and circle at  $P$  and  $p$  respectively will meet the axis in the same point.

Now in the circle the tangent at  $p$  is manifestly parallel to  $qq'$ ,

$$\text{and } NQ : Nq :: mP : mp,$$

$\therefore$  the tangent at  $P$  is parallel to  $QQ'$ .

(3.) If  $Cq$  be joined, since the angle  $Cqo$  is a right angle, and  $Co$  is perpendicular to  $qq'$ ,

$$\therefore Cv : Cq :: Cq : Co, \quad (\text{Euclid, VI. 8 Cor.})$$

or, since  $Cq = Cp$ ,

$$Cv : Cp :: Cp : Co.$$

E

$$\begin{aligned}
 &\text{But } Cv : Cp :: CV : CP, \\
 &\text{and } Cp : Co :: CP : CO, \\
 &\therefore CV : CP :: CP : CO, \\
 &\therefore CO \cdot CV = CP^2.
 \end{aligned}$$

COR. From this it is manifest that if any number of chords be drawn parallel to each other in an ellipse, their middle points will all lie on the line drawn from the centre to the point where the tangent parallel to the chord meets the ellipse.

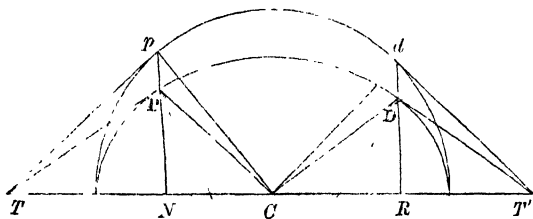
DEF. The line  $PCP'$  drawn through the centre of an ellipse and meeting the curve in  $P$  and  $P'$ , is called a *Diameter*.

The diameter consequently bisects all chords parallel to the tangents at its extremities; and the tangents at the extremities of any chord will intersect the diameter corresponding to that chord in the same point. ▴

36. DEF. If  $CD$  be drawn parallel to the tangent at  $P$ , then  $CD$  is said to be *conjugate* to  $CP$ .

### PROP. XIX.

(In the ellipse if  $CD$  be conjugate to  $CP$ , then will  $CP$  be conjugate to  $CD$ .)



Draw the ordinates  $PN$ ,  $DR$ , and produce them to meet the auxiliary circle in the points  $p$ ,  $d$ .

Join  $CP$ ,  $Cp$ ;  $CD$ ,  $Cd$ ; and draw the tangents  $TP$ ,  $Tp$ ;  $T'D$ ,  $T'd$ .

Now, since  $CD$  is parallel to  $PT$ ,

∴ the triangle  $PNT$  is similar to the triangle  $DR C$ .

$$\therefore TN : CR :: PN : DR,$$

$$:: Np : Rd, \text{ (Prop. XIII. Cor.)}$$

∴  $Tp$  is parallel to  $Cd$ ,

∴ the angle  $pCd$  is a right angle,

∴  $Cp$  is parallel to  $T'd$ ,

∴ the triangle  $pCN$  is similar to the triangle  $dTR$ ,

$$\therefore NC : RT' :: Np : Rd,$$

$$:: NP : RD,$$

∴  $CP$  is parallel to  $DT'$ ,

∴  $CP$  is conjugate to  $CD$ .

COR. Since  $CRd$  and  $CNp$  are each similar to  $dRT'$ ,  
(*Euclid*, VI. 8)

∴ the triangle  $CRd$  is similar to the triangle  $CNp$ ,

and the side  $Cd =$  the side  $Cp$ ,

∴ the triangle  $CRd =$  the triangle  $CNp$  in all respects,

$$\therefore CN = Rd, \text{ and } CR = Np.$$

$$\text{Hence } DR : CN :: DR : Rd,$$

$$:: BC : AC;$$

$$\text{also } PN : CR :: PN : Np,$$

$$:: BC : AC.$$

# PROP. XX.

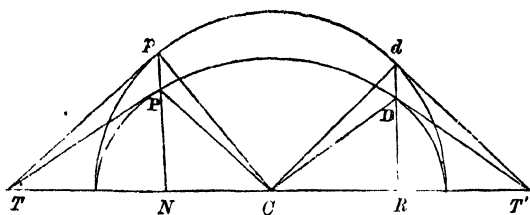
(37. If  $CP$  and  $CD$  be conjugate semi-diameters, and  $PN$ ,  $DR$  be the ordinates of the points  $P$  and  $D$ ; then

$$(1.) CN^2 + CR^2 = AC^2.$$

$$(2.) PN^2 + DR^2 = BC^2.$$

$$(3.) CP^2 + CD^2 = AC^2 + BC^2.)$$





Produce  $NP$ ,  $RD$  to meet the auxiliary circle in the points  $p$ ,  $d$ ; then

$$CN = Rd, \quad (\text{Prop. XIX. Cor.})$$

$$\begin{aligned} \therefore CN^2 + CR^2 &= Rd^2 + CR^2, \\ &= Cd^2, \\ &= CA^2. \end{aligned}$$

$$\text{Again, } PN : Np :: BC : AC,$$

$$\therefore PN^2 : Np^2 :: BC^2 : AC^2.$$

$$\text{So } DR^2 : Rd^2 :: BC^2 : AC^2,$$

$$\therefore PN^2 + DR^2 : Np^2 + Rd^2 :: BC^2 : AC^2;$$

$$\begin{aligned} \text{but } Np^2 + Rd^2 &= CR^2 + CN^2, \\ &= AC^2, \end{aligned}$$

$$\therefore PN^2 + DR^2 = BC^2,$$

$$\text{and } CN^2 + CR^2 = AC^2,$$

$$\therefore CP^2 + CD^2 = AC^2 + BC^2.$$

38. DEF. A line  $QV$  drawn parallel to the tangent at  $P$ , and meeting  $CP$  in  $V$ , is called an *Ordinate* to the diameter  $CP$ .

### PROP. XXI.

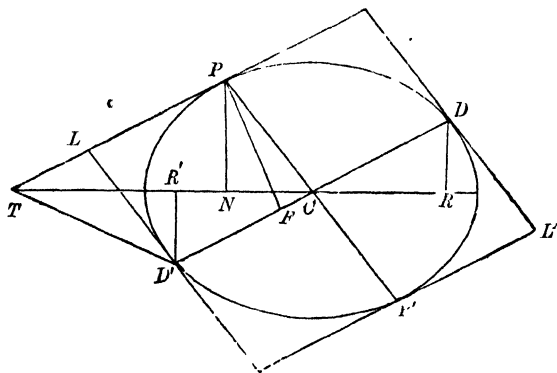
(If  $QV$  be any ordinate to the diameter  $PCP'$ , and  $CD$  be conjugate to  $CP$ ; then

$$QV^2 : PV \cdot P'V :: CD^2 : CP^2.)$$



diameters is equal to the rectangle contained by the axes of the ellipse.)

Let  $PCP'$ ,  $DCD'$  be a pair of conjugate diameters, and let a parallelogram be formed by drawing tangents at the points  $P$ ,  $P'$ ,  $D$ ,  $D'$ .



Let the tangent at  $P$  meet  $CA$  produced in  $T$ ; join  $D'T$ .

Draw the ordinates  $PN$ ,  $DR$ ,  $D'R'$ ; then since  $PT$  is parallel to  $CD'$ , the parallelogram  $PD'$  is double the triangle  $CTD'$ , and therefore equal to the rectangle contained by  $CT$  and  $D'R'$ .

Now  $D'R' : CN :: BC : AC$ , (*Prop. XIX. Cor.*)

$\therefore CT \cdot D'R' : CT \cdot CN :: BC : AC$ , (*Euclid, VI. 1*)

$:: BC \cdot AC : AC^2$ . (*Euclid, VI. 1.*)

But  $CT \cdot CN = AC^2$ ,

$\therefore CT \cdot D'R' = AC \cdot BC$ ,

$\therefore$  the parallelogram  $LL' = 4$  the parallelogram  $PD'$ ,

$= 4AC \cdot BC$ ,

$= AA' \cdot BB'$ .

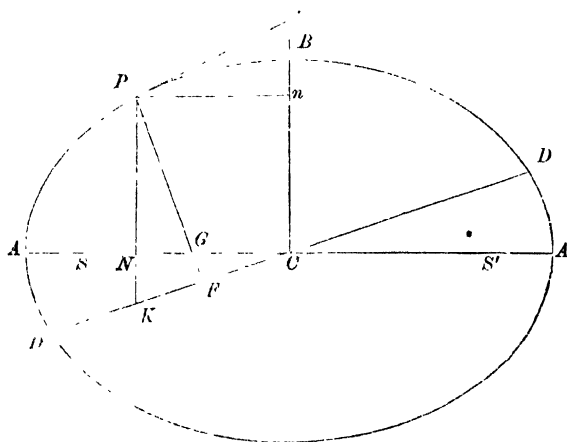
COR. If  $PF$  be drawn at right angles to  $DCD'$  meeting  $CD'$  in  $F$ ; then

$$\begin{aligned} PF \cdot CD' &= \text{area of parallelogram } PD', \\ &= AC \cdot BC. \end{aligned}$$

## PROP. XXIII.

(40. If  $CP$  and  $CD$  be conjugate diameters, and  $PF$  be drawn at right angles to  $CD$  meeting  $CA$  in  $G$ , then

$$PF \cdot PG = BC^2. \quad \bullet$$



Draw the ordinate  $PN$ , and produce it to meet  $CD'$  in  $K$ .

Also draw  $Pn$  at right angles to  $CB$ , and let the tangent at  $P$  meet  $CB$  produced in  $t$ .

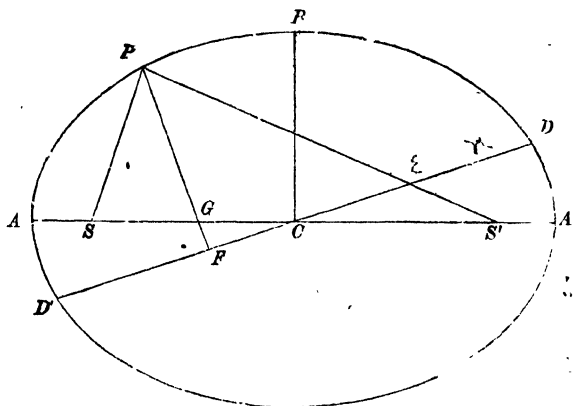
Now, since the angles at  $N$  and  $F$  are right angles, it is evident that a circle may be described about the quadrilateral figure  $NKFG$ ;

$$\begin{aligned} \therefore PG \cdot PF &= PN \cdot PK, \quad (\text{Euclid, III. 36 Cor.}) \\ &= Ct \cdot Cn, \\ &= BC^2. \quad (\text{Prop. XIV.}) \end{aligned}$$

## PROP. XXIV.

(41. If  $P$  be any point on the ellipse, and  $CD$  be conjugate to  $CP$ , then

$$SP \cdot S'P = CD^2.$$



Draw the normal  $PG$  and produce it to meet  $CD'$  in  $F$ ; then since  $CD'$  is parallel to the tangent at  $P$ ,

$\therefore PF$  is at right angles to  $CD'$ ,

$\therefore PF \cdot CD = AC \cdot BC$ , (*Prop. XXII. Cor.*)

and  $PF \cdot PG = BC^2 = BC \cdot BC$ , (*Prop. XXIII.*)

$\therefore CD : PG :: AC : BC$ . (1)

Again,  $SP : SG :: CA : CS$ , (*Prop. XI.*)

$S'P : S'G :: CA : CS$ . (*Prop. XI.*)

Compounding  $SP \cdot S'P : SG \cdot S'G :: CA^2 : CS^2$ ,  
 $\therefore SP \cdot S'P : SP \cdot S'P - SG \cdot S'G :: CA^2 : CA^2 - CS^2$ .

But  $SP \cdot S'P - SG \cdot S'G = PG^2$ , (*Euclid, VI. Prop. B*)

$\therefore SP \cdot S'P : PG^2 :: CA^2 : BC^2$ .

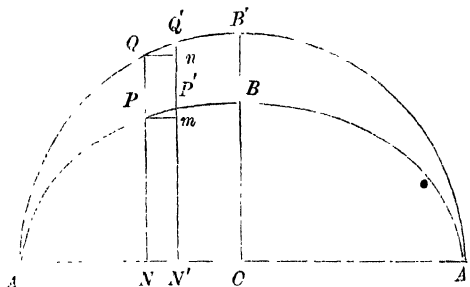
But from (1)  $CD^2 : PG^2 :: CA^2 : BC^2$ ,

$\therefore SP \cdot S'P = CD^2$ .

This proposition may also be very easily deduced from *Prop. XV.*

## PROP. XXV.

42. The area of the ellipse is to the area of the auxiliary circle as  $BC$  to  $AC$ . )



Let  $PN$  and  $P'N'$  be two ordinates of the ellipse near together.

Produce  $NP$ ,  $N'P'$ , to meet the auxiliary circle in  $Q$  and  $Q'$ .

Draw  $Pm$ ,  $Qn$ , perpendicular to  $Q'N'$ .

Then

the parallelogram  $PN'$  : the parallelogram  $Q'N'$  ::  $PN : QN$ ,  
::  $BC : AC$ .

And the same will be true for all the parallelograms that can be similarly described in the ellipse and auxiliary circle.

Hence the sum of all the parallelograms inscribed in the ellipse is to the sum of all the parallelograms inscribed in the circle as  $BC$  to  $AC$ .

And this holds however the number of parallelograms be increased.

But when the number of parallelograms is increased, and the breadth of each diminished indefinitely, the sum of the parallelograms inscribed in the ellipse will be equal to the area of the ellipse, and the sum of those inscribed in the circle to the area of the circle. Hence

the area of the ellipse : the area of the circle ::  $BC : AC$ .

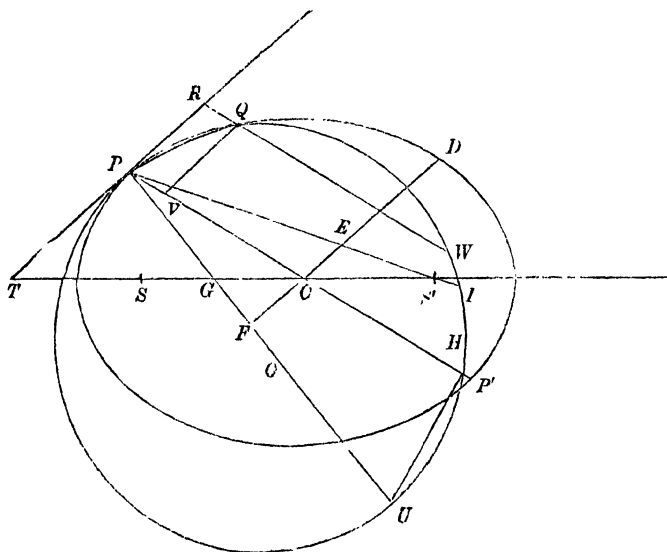
43. DEF. If with a point  $O$  on the normal at  $P$  as centre, and  $OP$  as radius, a circle be described touching the ellipse at  $P$ , and cutting it in  $Q$ ; then, when the point  $Q$  is made to approach indefinitely near to  $P$ , the circle is called the *Circle of Curvature* at the point  $P$ .

### PROP. XXVI.

(If  $PH$  be the chord of the circle of curvature at the point  $P$  of an ellipse, which passes through the centre; then

$$PH \cdot CP = 2CD^2.)$$

Let  $PT$  be the tangent, and  $PG$  the normal at the point  $P$ .



With centre  $O$ , and radius  $OP$ , describe a circle cutting the ellipse in the point  $Q$ .

Draw  $RQW$  parallel to  $CP$ , meeting the circle in  $W$ , and  $TP$  produced in  $R$ .

Also draw  $QV$  parallel to  $PR$ , meeting the diameter  $PP'$  in  $V$ ; then since  $RP$  touches the circle at  $P$ ,

$$\therefore RQ \cdot RW = PR^2, \text{ (Euclid, III. 36)}$$

$$\text{or } PV \cdot RW = QV^2.$$

$$\text{But } QV^2 : PV \cdot P'V :: CD^2 : CP^2, \text{ (Prop. XXI.)}$$

$$\therefore PV \cdot RW : PV \cdot P'V :: CD^2 : CP^2,$$

$$\text{or } RW : P'V :: CD^2 : CP^2.$$

Now, when the circle becomes the circle of curvature at  $P$ , the points  $R$  and  $Q$  move up to, and coincide with  $P$ , and the lines  $RW$  and  $PH$  become equal, while

$$P'V \text{ becomes equal to } PP', \text{ or } 2CP.$$

$$\text{Hence, } PH : 2CP :: CD^2 : CP^2,$$

$$\therefore PH \cdot CP : 2CP^2 :: 2CD^2 : 2CP^2,$$

$$\therefore PH \cdot CP = 2CD^2.$$

#### PROP. XXVII.

(If  $PU$  be the diameter of the circle of curvature at the point  $P$  of the ellipse, and  $PF$  be drawn at right angles to  $CD$ ; then

$$PU \cdot PF = 2CD^2.)$$

Since the triangle  $PHU$  is similar to the triangle  $PFC$ ,

$$\therefore PU : PH :: CP : PF,$$

$$\therefore PU \cdot PF = PH \cdot CP,$$

$$= 2CD^2. \text{ (Prop. XXVI.)}$$

#### PROP. XXVIII.

(If  $PI$  be the chord of the circle of curvature through the focus of the ellipse; then

$$PI \cdot AC = 2CD^2.)$$



Let  $PI$  meet  $CD$  in  $E$ ; then, since the triangles  $PIU$  and  $PEF$  are similar,

$$\therefore PI : PU :: PF : PE.$$

But  $PE = AC$ , (*Prop. XV. Cor.*)

$$\therefore PI : PU :: PF : AC,$$

$$\begin{aligned}\therefore PI \cdot AC &= PU \cdot PF, \\ &= 2CD^2. \text{ (} \textit{Prop. XXVII.} \text{)}\end{aligned}$$

### PROP. XXIX.

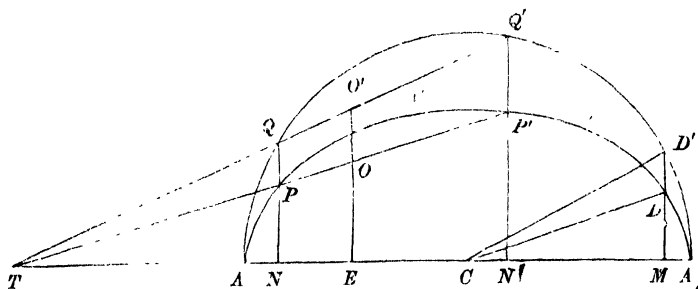
(44. If two chords of an ellipse intersect one another, the rectangles contained by their segments are proportional to the squares of the diameters parallel to them.)

Let  $POP'$  be any chord drawn through the point  $O$ , and let  $CD$  be the semi-diameter parallel to it

Draw the ordinates  $NP, N'P', MD$  and produce them to meet the auxiliary circle in  $Q, Q', D'$ ; then

since  $NP : NQ :: N'P' : N'Q'$ , (*Prop. XIII. Cor.*)

it is evident that  $PP'$  and  $QQ'$  will meet the axis produced in the same point  $T$ .



Also since  $NP : NQ :: MD : MD'$ , (*Prop. XIII. Cor.*)

and  $TPP'$  is parallel to  $CD$ ,

$\therefore TQ Q'$  is parallel to  $CD'$ .

Draw  $EO$  parallel to  $NQ$  or  $N'Q'$ , and produce it to meet  $QQ'$  in  $O'$ ; then

$$\begin{aligned} PO : QO' &:: TO : TO', \\ \text{and } P'O : Q'O' &:: TO : TO', \\ \therefore PO \cdot P'O : QO' \cdot Q'O' &:: TO^2 : TO'^2, \\ &:: CD^2 : CD'^2, \\ &:: CD^2 : AC^2. \end{aligned}$$

Alternately,  $PO \cdot P'O : CD^2 :: QO' \cdot Q'O' : AC^2$ .

Again, if through the point  $O$  any other chord  $pOp'$  be drawn,

$$\text{since } EO : EO' :: BC : AC,$$

it is manifest that the corresponding chord  $qq'$  in the auxiliary circle will pass through the point  $O'$ ; and if  $Cd$  be the semi-diameter parallel to  $pp'$  we shall have as before,

$$pO \cdot p'O : Cd^2 :: qO' \cdot q'O' : AC^2.$$

But  $qO' \cdot q'O' = qO' \cdot q'O'$ , (*Euclid*, III. 35)

$$\therefore PO \cdot P'O : CD^2 :: pO \cdot p'O : Cd^2,$$

$$\text{or } PO \cdot P'O : pO \cdot p'O :: CD^2 : Cd^2.$$

The same result may be shown to be true when the point  $O$  is without the ellipse.

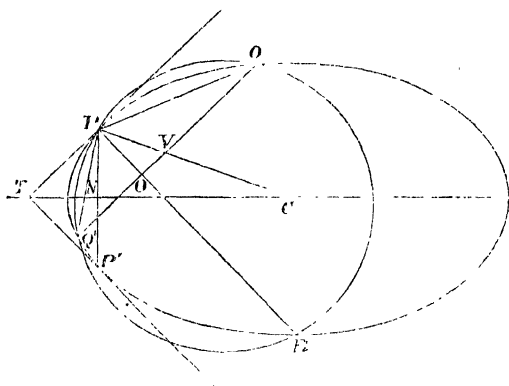
### PROP. XXX. ✓

(If  $QVQ'$  be any ordinate to the diameter  $CP$ , the circle described through the three points  $P, Q, Q'$  will intersect the ellipse in a fourth point, which depends only upon the position of  $P$ .)

Draw the ordinate  $PN$ , and produce it to meet the ellipse in  $P'$ ; then, since, if  $NT$  be the subtangent of either  $P$  or  $P'$ ,

$$CT \cdot CN = AC^2, \text{ (Prop. IX.)}$$

therefore the tangents at  $P$  and  $P'$  will meet the major axis produced in the same point  $T$ .



Draw  $PR$  parallel to  $TP'$ , meeting the ellipse in  $R$ , and  $QQ'$  in  $O$ ; then if  $CD$  and  $CD'$  be drawn parallel respectively to  $TP$  and  $TP'$ , meeting the ellipse in  $D$  and  $D'$ ,

$$PO \cdot OR : QO \cdot OQ' :: CD'^2 : CD^2. \quad (\text{Prop. XXIX.})$$

But  $CD' = CD$ , since  $CP' = CP$ ,

$$\therefore PO \cdot OR = QO \cdot OQ'.$$

Hence, by the converse of *Euclid* III. *Prop.* 22, the point  $R$  is on the circle which passes through  $P$ ,  $Q$ ,  $Q'$ .

COR. When the point  $V$  is brought indefinitely near to  $P$ ,  $QQ'$  coincides with the tangent to the ellipse at  $P$ , and becomes also a tangent to the circle at  $P$  since  $Q$  and  $Q'$  are indefinitely near to each other. The circle therefore becomes the circle of curvature at the point  $P$ .

Hence, if  $PR$  be drawn parallel to the tangent at  $P'$ , or be equally inclined to the axis with  $PT$ , it will meet the ellipse in the point where the circle of curvature at  $P$  intersects the ellipse.

## PROBLEMS ON THE ELLIPSE.

1. In what position of  $P$  is the angle  $SPS'$  greatest?
2. The latus rectum is a third proportional to the axis major and axis minor.
3. Construct on the axis minor as base, a rectangle which shall be to the triangle  $SLS'$  in the duplicate ratio of the major axis to the minor axis,  $L$  being the extremity of the latus rectum.
4. If a series of ellipses be described having the same major axis; the tangents at the extremities of their latera recta will all meet the minor axis in the same point.
5. Find the locus of the centres of all the ellipses having the same focus, and their major axes of the same length, and touching a given straight line.
6. Given the foci, it is required to describe an ellipse touching a given straight line.
7. If  $PT$  be a tangent to an ellipse, meeting the axis in  $T$ , and  $AP, A'P$ , be produced to meet the perpendicular to the major axis through  $T$  in  $Q$  and  $Q'$ , then  $QT = Q'T$ .
8. If the angle  $SBS'$  be a right angle, prove that  $CA^2 = 2CB^2$ .
9. If  $CP$  be a semi-diameter, and  $AQO$  be drawn parallel to  $CP$  meeting the curve in  $Q$ , and  $CB$  produced in  $O$ , then  $CP^2 = AO \cdot AQ$ .
10. If  $AB, CD$ , which are not parallel, make equal angles with either axis, the lines  $AC, BD$ , as also  $AD, BC$ , will make equal angles with either axis.

11.  $PSp$  is any focal chord.  $PA$  and  $pA$  are produced to meet the directrix in  $Q$  and  $q$ . Prove that the angle  $QSp$  is a right angle.

12. If a circle be described touching the axis major in one focus, and passing through one extremity of the axis minor;  $AC$  will be a mean proportional between the diameter of this circle and  $BC$ .

13. If  $PQQ'P'$  be a chord of the auxiliary circle, and a circle be described on the minor axis as diameter, cutting the chord in  $Q$  and  $Q'$ , then  $PQ \cdot P'Q = CS^2$ .

14. If  $PG$  be the normal at  $P$ , and  $GL$  be drawn at right angles to  $SP$ , then  $PL = \frac{1}{2}$  latus rectum.

15. The sum of the squares of the normals at the extremities of conjugate diameters is constant.

16. If on the normal at  $P$ ,  $PQ$  be taken equal to the semi-conjugate diameter  $CD$ , the locus of  $Q$  is a circle whose radius is  $AC - BC$ .

17. Find the locus of the intersection of a pair of tangents at right angles to each other.

18.  $P$  is any point on an ellipse. To any point  $Q$  on the curve draw  $AQ$ ,  $A'Q$ , meeting  $NP$  in  $R$  and  $S$ , and prove that  $NR \cdot NS = NP^2$ .

19. If  $PG$  be a normal, and  $GL$  perpendicular to  $SP$ , the ratio of  $GL$  to  $PN$  is constant.

20. If  $NP$  produced meet the tangent at the extremity of the latus rectum in  $Q$ , then  $QN = PS$ .

21. In an ellipse the tangent at any point makes a greater angle with the focal distance than with the perpendicular on the directrix.

22. A diameter of an ellipse, parallel to the tangent at any point, meets the focal distances of the point, and from the points of intersection lines are drawn perpendicular to the focal distances. Prove that these lines intersect in the axis minor.

23. The subnormal is a third proportional to  $CT$  and  $BC$ .

24. If  $PN$  be the ordinate of  $P$ , prove that  $NY : NY' :: PY : PY'$ . (See fig. Prop. XV.)

25. If from  $C$  lines be drawn parallel and perpendicular to the tangent at  $P$ , they inclose a part of one of the focal distances of that point equal to the other.

26. If  $P$  be a fixed point on an ellipse, and  $QQ'$  an ordinate to  $CP$ , the circle  $QPQ'$  will meet the ellipse in a fixed point.

27.  $P$  is any point on an ellipse. Draw  $PP'$  parallel to the axis major, and through  $P'$  draw  $P'Q, P'Q'$ , making equal angles with the major axis. Join  $QQ'$ ; then  $QQ'$  is parallel to the tangent at  $P$ .

28. What parallelogram circumscribing an ellipse has the least area?

29. When is the square of the sum of conjugate diameters least?

30. Given the axes of an ellipse, and the position of one focus, and of one point in the curve, give a geometrical construction for finding the centre.

31. If lines drawn through any point of an ellipse to the extremities of any diameter meet the conjugate  $CD$  in  $M$  and  $N$ , then  $CM \cdot CN = CD^2$ .

32. If  $CP$  and  $CD$  be conjugate, prove that

$$(SP - AC)^2 + (SD - AC)^2 = SC^2.$$

33. If  $CP$  and  $CD$  be conjugate, and  $BP, BD$  be joined, as also  $AD, A'P$ , these latter meeting in  $O$ , then  $BDO P$  is a parallelogram. When is the area greatest?

34. If  $PSp, QCq$  be two parallel chords through the focus and centre of an ellipse, prove that

$$SP \cdot Sp : CQ \cdot Cq :: BC^2 : AC^2.$$

35. If the tangent at the vertex  $A$  cut any two conjugate diameters in  $T$  and  $t$ , then  $AT \cdot At = AC^2$ .  ~~$AC^2$~~   $AC^2$ .

36. If the tangents at three points  $P, Q, R$  intersect in  $R, Q, P$ , prove that

$$PR \cdot P, Q \cdot Q, R = PQ \cdot R, Q \cdot P, R.$$

37. If a circle be described touching  $SP$ ,  $S'P$  produced, and the major axis of the ellipse, find the locus of the centre.

38. If from the extremities of the axes of an ellipse any four parallel lines be drawn, the points in which they cut the curve are the extremities of conjugate diameters.

39. If two equal and similar ellipses have a common centre, the points of intersection are at the extremities of diameters at right angles to one another.

40. If  $PSQ$  be a focal chord, and  $X$  the foot of the directrix,  $XP$  and  $XQ$  are equally inclined to the axis.

41.  $OP$ ,  $OQ$  are tangents to an ellipse, and  $PQ$  is produced to meet the directrices in  $R$ ,  $R'$ , prove that

$$RP \cdot R'P : RQ \cdot R'Q :: OP^2 : OQ^2.$$

42.  $NPQ$  is a common ordinate to the ellipse and auxiliary circle.  $PR$ ,  $QR$  are normals at  $P$  and  $Q$  intersecting in  $R$ . The locus of  $R$  is a circle whose radius is  $AC + BC$ .

43. If the conjugate to  $CP$  meet  $SP$ ,  $S'P$ , or these produced in  $E$ ,  $E'$ ; then  $SE = S'E'$ , and the circles circumscribing  $SCE$ ,  $S'CE'$  are equal.

44. The locus of the middle points of all focal chords in an ellipse is a similar ellipse.

45. The circle described about the triangle  $SBS'$  will cut the minor axis in the centre of the circle of curvature at  $B$ .

46. The locus of the centre of the circle inscribed in the triangle  $SPS'$  is an ellipse.

47. If a circle be described intersecting an ellipse in four points, and chords be drawn through the points of intersection, diameters parallel to the chords will be equal.

48. An ellipse slides between two lines at right angles to each other, find the locus of its centre.

49. If from the focus  $S$  perpendiculars be drawn upon the conjugate diameters  $CP$ ,  $CD$ , these perpendiculars produced backward will intersect  $CD$  and  $CP$  in the directrix.

50. Find the point at which the radius of curvature is a mean proportional between the major and minor axes.

51. The circle of curvature at a point, where the conjugate diameters are equal, meets the ellipse again at the extremity of the diameter.

52. The locus of the intersection of lines drawn from  $A, A'$  at right angles to  $AP, A'P$  is an ellipse.

53. If a quadrilateral figure be inscribable in two ellipses whose major axes are parallel or perpendicular, any two of its opposite angles will be equal to two right angles.

54. If  $CN, NP$  are the abscissa and ordinate of a point  $P$  on a circle whose centre is  $C$ , and  $NQ$  be taken equal to  $NP$ , and be inclined to it at a constant angle, the locus of  $Q$  is an ellipse.

55. If two ellipses having the same major axes can be inscribed in a parallelogram, the foci will be on the corners of an equiangular parallelogram.

56. Two ellipses, whose major axes are equal, have a common focus. Prove that they intersect in two points only.

57. A circle described about the triangle  $SPS'$  cuts the minor axis in  $R$  on the opposite side to  $P$ . Prove that  $SR$  varies as the normal  $PG$ .

58. If  $r$  and  $R$  be the radii of the circles inscribed in and about the triangle  $SPS'$ , prove that  $R \cdot r$  varies as  $SP \cdot S'P$ .

59. The circle described upon  $PG$  as diameter cuts  $SP, S'P$  in  $K$  and  $L$ . Prove that  $KL$  is bisected by  $PG$ , and is perpendicular to it.

60. If from  $S'$  a line be drawn parallel to  $SP$ , it will meet  $SY$  in the circumference of a circle.

61.  $T$  and  $t$  are the points where the tangent at  $P$  meets the axes.  $CP$  is produced to meet in  $L$  the circle described about the triangle  $T Ct$ ; prove that  $PL$  is half the chord of the circle of curvature at  $P$  in the direction of  $C$ , and that  $CP \cdot CL$  is constant.

62. About the triangle  $PQR$  an ellipse is described, having its centre at the point where the lines drawn from  $P, Q, R$  to the middle points of the opposite sides meet.  $CP, CQ, CR$  are produced to meet the ellipse in  $P', Q', R'$ . Prove that



the tangents at  $P'$ ,  $Q'$ ,  $R'$  form a triangle similar to  $PQR$ , and four times as large.

63. Lines from  $Y$  and  $Y'$  perpendicular to the major axis cut the circles on  $SP$ ,  $S'P$  as diameters in  $I$  and  $J$ . Prove that  $IS$ , and  $JS'$ , when produced, intersect  $BC$  in the same point.

64. If from the ends of any diameter chords be drawn to any point in the ellipse, the diameters parallel to these chords will be conjugate.

65. If  $T$  be the angle between tangents at the extremities of a focal chord, and  $O$  the angle subtended by the chord at the other focus, then

$$2T + O = 2 \text{ right angles.}$$

66. The acute angles which  $SP$ ,  $SQ$  make with the tangents are complementary. Prove that  $BC^2$  is a mean proportional between the areas of the triangles  $SPS'$ ,  $SQS'$ . Also, show that the problem is impossible unless  $BC < CS$ .

67. A series of ellipses have their equal conjugate diameters of the same magnitude. One of these diameters is fixed and common, while the other varies. The tangents drawn from any point in the fixed diameter produced will touch the ellipses in points situated on a circle.

68. If on the longer side of a rectangle as major axis an ellipse be described, passing through the intersection of the diagonals, and lines be drawn from any point of the ellipse exterior to the rectangle to the ends of the remote side, they will divide the major axis into segments, which are in geometric progression.

69. From any point  $P$  of an ellipse  $PQ$  is drawn at right angles to  $SP$  meeting the diameter conjugate to  $CP$  in  $Q$ . Prove that  $PQ$  varies inversely as the perpendicular from  $P$  on the major axis.

70. In an ellipse  $SQ$  and  $S'Q$ , drawn at right angles to a pair of conjugate diameters, intersect in  $Q$ . Prove that the locus of  $Q$  is a concentric ellipse.

## CHAPTER III.

### THE HYPERBOLA.

45. DEF. The *Hyperbola* is the curve traced out by a point which moves in such a manner that its distance from a given fixed point continually bears the same ratio, *greater than unity*, to its distance from a given fixed line. (*See Introduction.*)

#### PROP. I.

(The focus and directrix of a hyperbola being given, to find any number of points on the curve.)

Let  $S$  be the focus, and  $MX$  the directrix.

Draw  $SX$  at right angles to the directrix, and divide  $SX$  in the point  $A$ , so that  $SA$  may be to  $AX$  in the given fixed ratio, greater than unity; then

$A$  is a point on the curve.

On  $SX$  produced take a point  $A'$ , such that

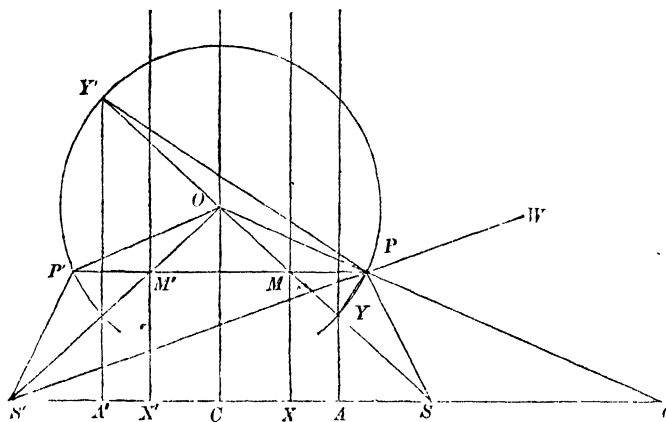
$$SA' : A'X :: SA : AX;$$

then  $A'$  will also be a point on the curve.

On the directrix take *any* point  $M$ ; and through  $S$  and  $M$  draw the line  $SYMY'$ , meeting  $AY$  and  $A'Y'$ , drawn at right angles to  $AA'$ , in the points  $Y$  and  $Y'$ ;

On  $YY'$  as diameter describe a circle, and draw  $PMP'$  parallel to  $AA'$ , cutting the circle in the points  $P$  and  $P'$ ;

$P$  and  $P'$  will be points on the hyperbola.



Join  $PY, PY', SP$ ; then since

$$SY : YM :: SA : AX, \quad (\text{Euclid, VI. 2.})$$

$$\text{and } SY' : Y'M :: SA' : A'X, \quad (\text{Euclid, VI. 2.})$$

$$\therefore SY : YM :: SY' : Y'M;$$

or, alternately,  $SY : SY' :: YM : Y'M$ ,

and the angle  $YPY'$  in a semicircle is a right angle,

$$\therefore PY \text{ bisects the angle } SPM,*$$

$$\therefore SP : PM :: SY : YM,$$

$$:: SA : AX.$$

So we may show that

$$SP' : P'M :: SY' : Y'M,$$

\* For, if not, make the angle  $YPm$  equal to  $YPS$ ; then

$$SY : Ym :: SP : Pm, \quad (\text{Euclid, VI. 3})$$

and, since, if  $PY$  bisect  $SPm$ ,  $PY'$  being at right angles to  $PY$ , also bisects the angle between  $MP$  and  $SP$  produced;

$$\therefore SY' : Y'm :: SP : Pm, \quad (\text{Euclid, VI. A.})$$

$$\text{Hence } SY : Ym :: SY' : Y'm,$$

$$\text{or } SY : SY' :: Ym : Y'm;$$

$$\therefore \text{the points } M \text{ and } m \text{ coincide.}$$

$$\therefore SA : AX,$$

$\therefore P$  and  $P'$  are points on the curve.

In the same way, by taking other points on the directrix, we may obtain as many more points on the curve as we please.

COR. 1. Since, corresponding to every point  $P$  on the curve, there is a point  $P'$  situated in precisely the same manner with respect to  $A'Y'$  as  $P$  is with respect to  $AY$ , it is clear that if we make  $A'S'$  equal to  $AS$ , and  $A'X'$  equal to  $AX$ , and draw  $X'M'$  at right angles to  $AX'$ , the curve could be equally well described with  $S'$  as focus and  $M'X'$  as directrix.

The hyperbola is therefore symmetrical, not only with respect to the line  $AA'$ , but also with respect to the line  $OC'$  drawn through the middle point of  $YY'$  at right angles to and bisecting  $AA'$ .

COR. 2. The line  $OP$  produced will bisect the angle  $SPW$  between  $SP$  and  $S'P$  produced.

Produce  $OP$  and  $S'S$  to meet in  $G$ . Produce  $PM$  to meet  $X'M'$  in  $M'$ , and draw  $OS'$  passing through the point  $M'$ ; then

$$SP : PM :: S'P : PM',$$

or, alternately,  $SP : S'P :: PM : PM'$ . (1.)

$$\text{Again, } SG : PM :: S'G : PM',$$

or, alternately,  $SG : S'G :: PM : PM'$ . (2.)

$\therefore$  from (1) and (2)

$$SP : S'P :: SG : S'G.$$

$\therefore PG$  bisects the angle  $SPW$ . (*Euclid*, VI. A.)

It will be shown hereafter (*Prop.* IX.) that the normal to the hyperbola at the point  $P$  also bisects the angle  $SPW$ . Hence the hyperbola and circle have the same tangent at the point  $P$ . The hyperbola will consequently touch all the infinite series of circles which can be described in the same manner as the one in the figure, by taking different points on the directrix.

## PROP. II.

(46. If  $C$  be the middle point of  $AA'$ ; then  $CA$  is a mean proportional between  $CS$  and  $CX$ , )

or  $CS \cdot CX = CA^2$ . (See *fig. Prop. III.*)

Since  $SA' : A'X :: SA : AX$ .

Alternately,  $SA' : SA :: A'X : AX$ ,

$\therefore SA' - SA : SA :: A'X - AX : AX$ ,

or  $AA' : SA :: XX' : AX$ ,

$\therefore AA' : XX' :: SA : AX$ ,

or  $CA : CX :: SA : AX$ . (1.)\*

Again,  $SA' : SA :: A'X : AX$ .

$\therefore SA' + SA : SA :: A'X + AX : AX$ ,

or  $SS' : SA :: AA' : AX$ .

Alternately,  $SS' : AA' :: SA : AX$ ,

or  $CS : CA :: SA : AX$ . (2.)

Hence, from (1) and (2),

$CA : CX :: CS : CA$ ,

$\therefore CA^2 = CX \cdot CS$ .

Or  $CA$  is a mean proportional between  $CS$  and  $CX$ .

COR. Since the three lines  $CS$ ,  $CA$ ,  $CX$ , are proportional, therefore, by the definition of duplicate ratio, and *Euclid*, VI. 20 *Cor.*,

$CS : CX :: CS^2 : CA^2$ . (3.)

## PROP. III.

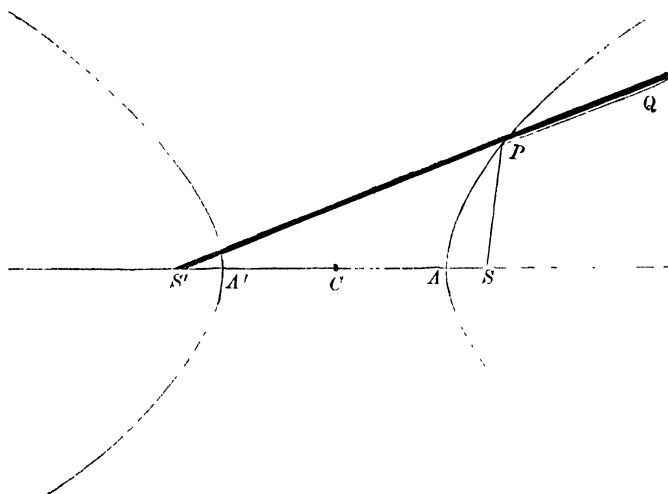
47. (If  $P$  be any point on the hyperbola, and  $S$  be the focus nearer to  $P$ ; then

$SP - PM = AA'$ .)

Since  $SP : PM :: SA : AX$ ,

\* N.B. The results (1), (2), (3), should be remembered, as they will be frequently referred to.





If the loose end of the string be now fastened to the focus  $S$ , and the rod being initially placed in the position  $S'S$ , be made to revolve round  $S'$ , while the string is kept constantly stretched by means of the point of a pencil at  $P$ , in contact with the bar; since  $S'P$  and  $SP$  are always increasing by the same amount, viz. the length of the portion of the string that removes itself from the bar, between any two positions of  $P$ , the difference between  $S'P$  and  $SP$  will be constantly the same, and the point  $P$  will trace out the hyperbola.

Another perfectly similar branch may be described in the same manner by making the bar revolve round  $S$  as centre.

In this case  $SP - S'P$  will be equal to  $AA'$ .

The curve, therefore, consists of two similar branches, which recede indefinitely both from the line  $AA'$ , and also from the line  $BCB'$  drawn bisecting  $AA'$  at right angles. (See *fig. Prop. IV.*)

48. If  $BC$  be taken of such a length that

$$BC^2 = CS^2 - CA^2,$$

and  $CB'$  be made equal to  $CB$ , then  $AA'$  and  $BB'$  are called respectively the *Transverse* and *Conjugate Axes*.

The line  $BCB'$  does not meet the hyperbola, and the reason of its being introduced will be seen further on.

If the conjugate and transverse axes are equal, the hyperbola is said to be *rectangular* or *equilateral*.

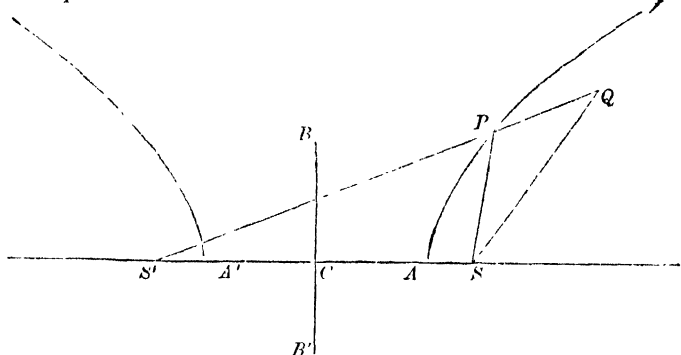
The property of the hyperbola, which we have just investigated, viz. that the difference between  $SP$  and  $S'P$  is constant, is sometimes taken as the definition of the curve. (See Chapter II. Art. 25.)

Also as in the ellipse, if  $SL$  be the semi-latus rectum, it may be proved that

$$SL \cdot AC = BC^2.$$

#### PROP. IV.

49. (The difference of the distances of any point from the foci of a hyperbola will be greater or less than  $AA'$ , according as the point is on the concave or convex side of the curve.)

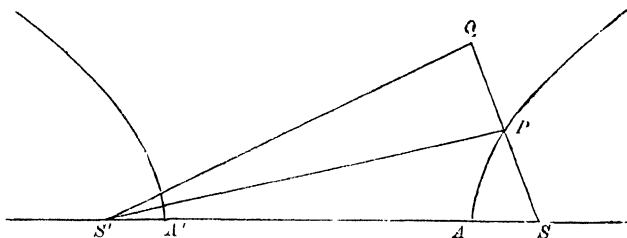


(1.) Let  $Q$  be a point on the concave side of the hyperbola.

Join  $SQ$ ,  $S'Q$ , and let  $S'Q$  meet the curve in  $P$ ; join  $SP$ ; then

$$\begin{aligned} \text{since } S'Q &= S'P + PQ, \\ \text{and } SQ &< SP + PQ, \end{aligned}$$





$$\begin{aligned}\therefore S'Q - SQ &> S'P - SP. \\ \text{but } S'P - SP &= AA', \\ \therefore S'Q - SQ &> AA'.\end{aligned}$$

(2.) Let  $Q$  be a point on the convex side of the curve, nearer to  $S$  than  $S'$ ; join  $SQ$ ,  $S'Q$ , and let  $SQ$  meet the curve in  $P$ ; join  $S'P$ ; then

$$\begin{aligned}S'Q &< S'P + PQ, \\ \text{and } SQ &= SP + PQ, \\ \therefore S'Q - SQ &< S'P - SP, \\ \text{but } S'P - SP &= AA', \\ \therefore S'Q - SQ &< AA',\end{aligned}$$

so if  $Q$  be nearer to  $S'$  than  $S$ , we can show that

$$SQ - S'Q < AA';$$

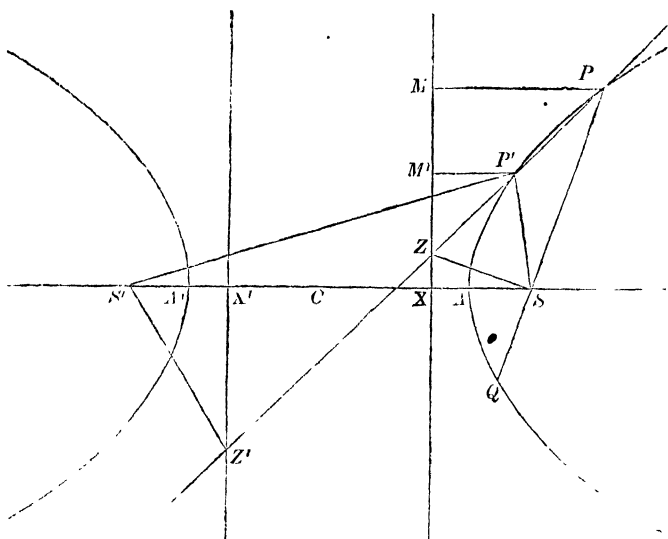
COR. Conversely a point will be on the concave or convex side of the hyperbola, according as the difference of its distances from the foci is greater or less than  $AA'$ .

50. DEF. If a point  $P'$  be taken on the hyperbola near to  $P$  (see *fig. Prop. V.*), and  $PP'$  be joined, the line  $PP'$  produced, in the limiting position which it assumes when  $P'$  is made to approach indefinitely near to  $P$ , is called the *Tangent* to the hyperbola at the point  $P$ .

#### PROP. V.

(If the tangent to the hyperbola at any point  $P$  meet the directrix in the point  $Z$ , and if  $S$  be the focus corresponding to the directrix on which  $Z$  is situated, then  $SZ$  will be at right angles to  $SP$ .)

# CONIC SECTIONS.



Let  $P'$  be a point on the curve near to  $P$ .

Draw the chord  $PP'$ , and produce it to meet the directrix in  $Z$ ; join  $SZ$ .

Draw  $PM$ ,  $P'M'$  at right angles to the directrix, and join  $SP$ ,  $SP'$ .

Produce  $PS$  to meet the hyperbola in  $Q$ ; then since the triangles  $ZMP$ ,  $ZM'P'$  are similar,

$$\begin{aligned}\therefore ZP : ZP' &:: MP : M'P', \\ &:: SP : SP' .\end{aligned}$$

$\therefore SZ$  bisects the angle  $P'SQ$ . (*Euclid*, VI. A.)

Now, when  $P'$  is indefinitely near to  $P$ , and  $PP'$  becomes the tangent at the point  $P$ , the angle  $PSP'$  becomes indefinitely small, while the angle  $QSP'$  approaches two right angles; and therefore the angle  $ZSP'$ , being half of the angle  $P'SQ$ , becomes ultimately a right angle.

Hence, when  $PZ$  becomes the tangent at the point  $P$ , the angle  $ZSP$  is a right angle,

or  $SZ$  is perpendicular to  $SP$ .

COR. 1. Conversely, if  $SZ$  be drawn at right angles to  $SP$ , meeting the directrix in  $Z$ , and  $PZ$  be joined,  $PZ$  will be the tangent at  $P$ .

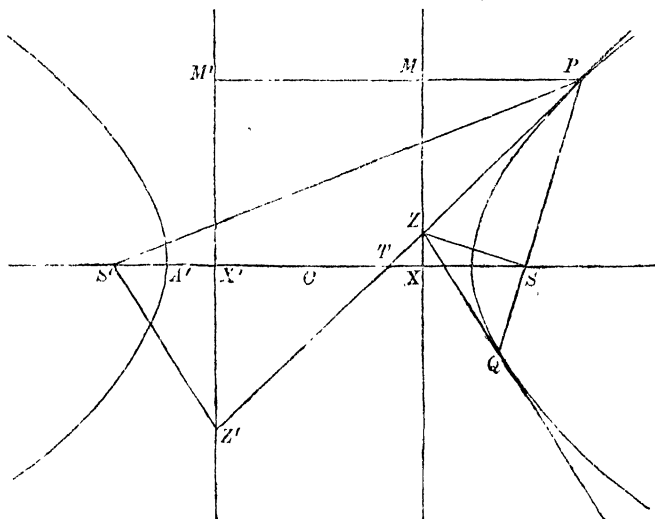
COR. 2. If  $PZ$  be produced to meet the other directrix in  $Z'$ , and  $S'Z'$  be joined; then

$S'Z'$  is at right angles to  $S'P$ .

COR. 3. The tangents at the extremities of the latus rectum, or double ordinate through the focus, meet the axis in the point  $X$ .

### PROP. VI.

(The tangent to the hyperbola at any point  $P$  makes equal angles with the focal distances  $SP$  and  $S'P$ .)



Let the tangent at  $P$  meet the directrices in  $Z$  and  $Z'$ .

Draw  $PMM'$  at right angles to the directrices meeting them in  $M$  and  $M'$  respectively; join  $SZ$ ,  $S'Z'$ ; then

$$SP : PM :: S'P : PM'.$$

And since the triangles  $ZMP$ ,  $Z'M'P$  are similar,

$$PM : PZ :: PM' : PZ',$$

$$\therefore SP : PZ :: S'P : PZ'. \quad (\text{Ex æquali.})$$

Now in the triangles  $SPZ$ ,  $S'PZ'$  because the sides about the angles  $SPZ$ ,  $S'PZ'$  are proportional, and the angles  $PSZ$ ,  $P'S'Z'$  are equal, being right angles, and the angles  $SZZP$ ,  $S'Z'P$  are each less than a right angle,

$\therefore$  the triangles  $SPZ$ ,  $S'PZ'$  are similar. (*Euclid*, VI. 7)

$\therefore$  the angle  $SPZ = S'PZ'$ .

### PROP. VII.

(The tangents at the extremities of a focal chord intersect in the directrix)

Let  $PSQ$  be a focal chord, and let the tangent at  $P$  meet the directrix in  $Z$ . Join  $SZ$ ; then

the angle  $ZSP$  is a right angle, (*Prop. V.*)

And  $\therefore$  also the angle  $ZSQ$  is a right angle,

$\therefore ZQ$  is the tangent at  $Q$ . (*Prop. V. Cor. 1.*)

Or the tangents at the extremities of a focal chord intersect in the directrix.

### PROP. VIII.

51. (If the tangent at  $P$  meet the transverse axis in  $T$ , and  $PN$  be the ordinate of the point  $P$ ; then

$$CT \cdot CN = CA^2. \quad )$$

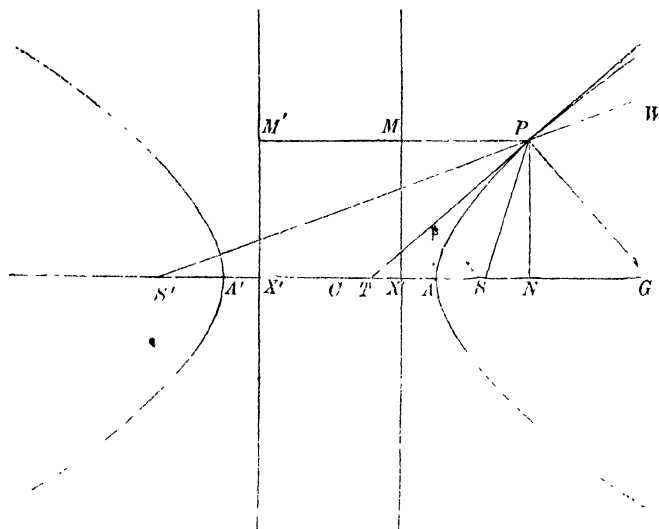
Draw  $PMM'$  at right angles to the directrices meeting them in  $M$  and  $M'$ . Join  $SP$ ,  $S'P$ ; then

since  $PT$  bisects the angle  $SPS'$ , (*Prop. VI.*)

$\therefore S'T : ST :: S'P : SP$ , (*Euclid*, VI. 3.)

$$:: PM' : PM,$$

$$:: X'N : XN.$$



$$\begin{aligned}
 \therefore S'T - ST : S'T + ST &:: X'N - XN : X'N + XN \\
 \text{or } 2CT : 2CS &:: 2CX : 2CN, \\
 \text{or } CT : CS &:: CX : CN \\
 \therefore CT \cdot CN &= CS \cdot CX, \\
 &= CA^2. \quad (\text{Prop. II.})
 \end{aligned}$$

52. DEF. The line  $PG$ , drawn at right angles to the tangent  $PT$ , is called the *Normal* to the hyperbola at the point  $P$ .

#### PROP. IX.

(If the normal to the hyperbola at the point  $P$  meet the transverse axis in the point  $G$ , and  $PN$  be the ordinate of the point  $P$ , then

$$NG : NC :: BC^2 : AC^2.)$$

Draw  $PMM'$  at right angles to the directrices, meeting them in  $M$  and  $M'$ , and produce  $S'P$  to  $W$ ; then since the angle  $TPG$  is a right angle,

$\therefore$  the angle  $WPG$  = the complement of the angle  $S'PT$ ,  
and the angle  $SPG$  = the complement of the angle  $SPT$ ;

but the angle  $S'PT =$  the angle  $SPT$ ,  
 $\therefore$  the angle  $WPG =$  the angle  $SPG$ ,  
 $\therefore PG$  bisects the angle  $SPW$ ,  
 $\therefore S'G : SG :: S'P : SP$ , (*Euclid*, VI. A.)  
 $:: PM' : PM$ ,  
 $:: X'N : XN$ ,  
 $\therefore S'G + SG : S'G - SG :: X'N + XN : X'N - XN$ ;  
or  $2CG : ES' :: 2CN : XX'$ .  
Alternately,  $2CG : 2CN :: SS' : XX$ ;  
or  $CG : CN :: CS : CX$ ,  
 $:: CS^2 : CA^2$  (*Prop. II. Cor.*)  
 $\therefore CG - CN : CN :: CS^2 - CA^2 : CA^2$ ;  
or  $NG : CN :: BC^2 : AC^2$ .

## PROP. X.

(If  $PN$  be the ordinate of any point  $I$  on the hyperbola,  
then

$$PN^2 : AN \cdot A'N :: BC^2 : AC^2.)$$

$$\text{For } NG : NC :: BC^2 : AC^2.$$

And rectangles of the same altitude are to one another as  
their bases, (*Euclid*, VI. 1.)

$$\therefore TN \cdot NG : TN \cdot NC :: BC^2 : AC^2,$$

$$\text{or } PN^2 : TN \cdot NC :: BC^2 : AC^2.$$

$$\text{But } TN \cdot CN = CN^2 - CT \cdot CN, \text{ (Euclid, II. 2.)}$$

$$= CN^2 - CA^2, \text{ (Prop. VIII.)}$$

$$= AN \cdot A'N, \text{ (Euclid, II. 6.)}$$

$$\therefore PN^2 : AN \cdot A'N :: BC^2 : AC^2.$$

## PROP. XI.

If the normal at any point  $P$  of an hyperbola meet the  
transverse axis in  $G$ ; then

$$SG : SP :: CS : CA.$$

Produce  $S'P$  to  $W$ ; then

since  $PG$  bisects the angle  $SPW$ , (*Prop. IX.*)

$$SG : S'G :: SP : S'P,$$

$$\therefore SG : S'G - SG :: SP : S'P - SP,$$

$$\text{but } S'P - SP = AA', \text{ (} \textit{Prop. III.} \text{)}$$

$$\text{and } S'G - SG = SS',$$

$$\therefore SG : SS' :: SP : AA',$$

$$\text{or } SG : SP :: SS' : AA',$$

$$\text{or } SG : SP :: CS : CA.$$

COR. Hence also,

$$S'G : S'P :: CS : CA.$$

## PROP. XII.

53. If from the foci  $S$  and  $S'$  of an hyperbola  $SY$  and  $S'Y'$  are drawn at right angles to the tangent at  $P$ , then  $Y$  and  $Y'$  are on the circumference of the circle described on  $AA'$  as diameter, and

$$SY \cdot S'Y' = BC^2.$$

Join  $SP$ ,  $S'P$ , and produce  $SY$  to meet  $S'P$  in  $W$ ; join  $CY$ ; then

since the angle  $SPY =$  the angle  $WPY$ , (*Prop. VI.*)

and the angle  $SYP =$  the angle  $WYP$ ,

and the side  $PY$  is common to the triangles  $SPY$ ,  $WPY$ .

$\therefore$  the triangle  $SPY = WPY$  in all respects,

$$\therefore SP = PW, \text{ and } SY = WY,$$

$$\therefore S'P - SP = S'W,$$

$$\text{but } S'P - SP = AA', \text{ (} \textit{Prop. III.} \text{)}$$

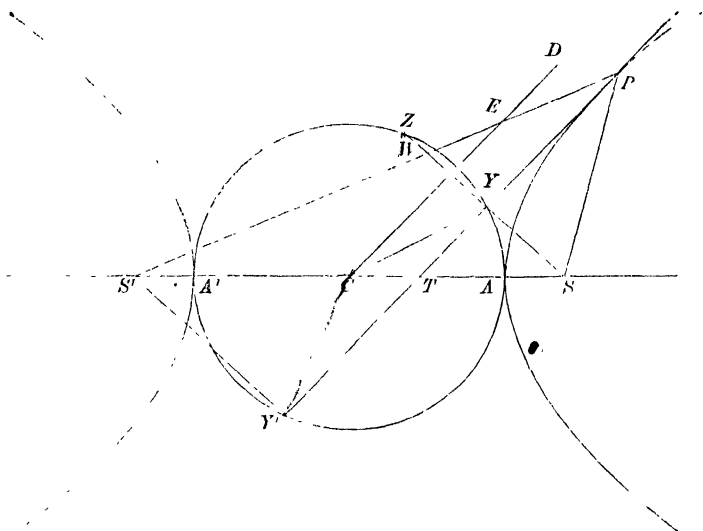
$$\therefore S'W = AA'.$$

Again,  $\therefore SC = CS'$ , and  $SY = WY$ ,

$$\therefore SC : CS' :: SY : WY,$$

$$\therefore CY \text{ is parallel to } S'W,$$

$$\therefore CY : SW :: CS : S'W,$$



$$\therefore CY = \frac{1}{2} S'W = CA;$$

so  $CY' = CA.$

$\therefore Y$  and  $Y'$  are points on the circumference of the circle described upon  $AA'$  as diameter.

Next, let  $SY$  be produced to meet this circle in  $Z$ , and join  $ZY'$ ; then

since the angle  $ZY'Y'$  is a right angle

$\therefore ZY'$  passes through the centre  $C$ ,

$\therefore$  the angle  $SCZ =$  the angle  $S'CY$ ,

$$\therefore SZ = S'Y',$$

$$\begin{aligned} \therefore SY \cdot S'Y' &= SY \cdot SZ, \\ &= SA \cdot SA', \text{ (Euclid, III. 36 Cor.)} \\ &= CS^2 - CA^2, \text{ (Euclid, II. 6.)} \\ &= BC^2. \end{aligned}$$

COR. If  $CD$  be drawn parallel to the tangent at  $P$  meeting  $S'P$  in  $E$ ; then

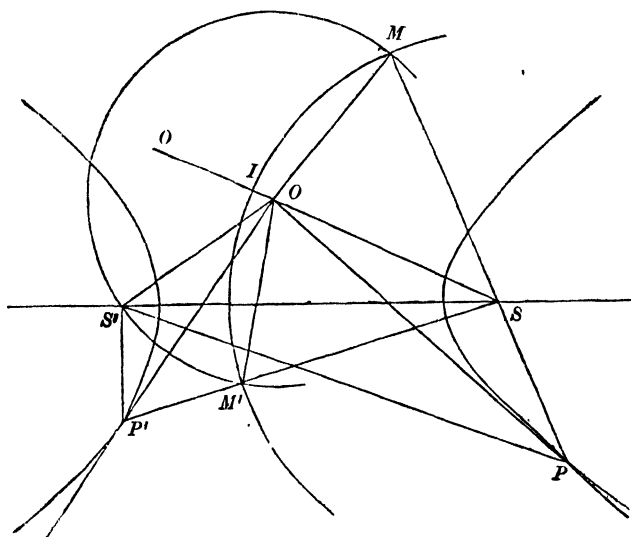
since  $CEPY$  is a parallelogram,

$$\therefore PE = CY = AC.$$



## PROP. XIII.

54. To draw a pair of tangents to an hyperbola from an external point  $O$ .



Of the foci  $S$  and  $S'$ , let  $S'$  be that which is nearer to  $O$ .

With centre  $S$  and radius equal to  $AA'$  describe a circle.

Join  $OS$ ,  $OS'$ ; and let  $SO$  or  $SO$  produced meet the circle in the point  $I$ .

Now if  $O$  be a point inside the circle  $MIM'$ , it is evident that  $OS'$  is greater than  $OI$ ; and if  $O$  be outside the circle,

since  $OS - OS' < AA'$  or  $SI$ , (*Prop. IV.*)

$\therefore OS - OS' < OS - OI$ ,

$\therefore OS' > OI$ .

With centre  $O$  and radius  $OS'$  describe another circle cutting the former in the points  $M$  and  $M'$ , which it will always do since  $OS'$  is greater than  $OI$ .

Join  $SM, SM'$ , and produce them to meet the hyperbola in the points  $P$  and  $P'$ .

Join  $OP, OP'$ ; these will be the tangents required.

Join  $S'P, S'P'$ ; then

since  $S'P - SP = AA' = SM$ ,

$\therefore S'P = PM$ .

And  $\therefore S'P, PO = MP, PO$ , each to each,

and  $OS' = OM$ ,

$\therefore$  the angle  $OPS' =$  the angle  $OPM$ ,

$\therefore OP$  is the tangent at  $P$ . (*Prop. VI.*)

So  $OP'$  is the tangent at  $P'$ .

The points of contact  $P$  and  $P'$  will be upon the same or opposite branches of the hyperbola according as  $SM$  and  $SM'$  require to be produced in the same or in opposite directions with respect to  $S$ , in order to intersect the hyperbola.

#### PROP. XIV.

If from a point  $O$  a pair of tangents,  $OP, OP'$  be drawn to an hyperbola, then the angles which  $OP$  and  $OP'$  subtend at either focus will be equal or supplementary according as the points of contact are in the same or opposite branches of the hyperbola.

Let the points  $P$  and  $P'$  be on opposite branches of the hyperbola.

Join  $SP, S'P; SP', S'P'$ .

Produce  $PS$  to  $M$ , making  $PM$  equal to  $PS'$ . Also from  $P'S$  cut off a part  $P'M'$  equal to  $P'S'$ .

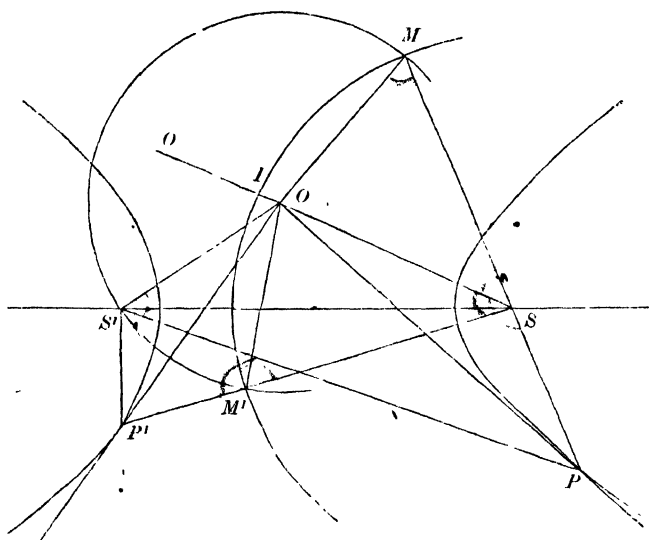
Join  $OM, OM'; OS, OS'$ .

Then since  $OP, PS' = OP, PM$ , each to each,

and the angle  $OPS' =$  the angle  $OPM$ , (*Prop. VI.*)

$\therefore OS' = OM$ ,

and the angle  $OS'P =$  the angle  $OMP$ .



So  $OS' = OM'$ ,  
 and the angle  $OS'P' =$  the angle  $OM'P'$ ,  
 $\therefore OM = OM'$ .  
 Again,  $\because SM = S'P - SP = AA'$ ,  
 and  $SM' = SP' - S'P' = AA'$ ,  
 $\therefore SM = SM'$ .  
 And  $\because OS, SM = OS, SM'$ , each to each,  
 and  $OM = OM'$ ,  
 $\therefore$  the angle  $OSM =$  the angle  $OSM'$ ,  
 and the angle  $OMS =$  the angle  $OM'S$ .  
 But  $OSM$  is the supplement of  $OSP$ ,  
 and  $OM'S$  the supplement of  $OM'P'$ ,  
 $\therefore OSM'$  is the supplement of  $OSP$ ,  
 and  $OMP$  the supplement of  $OM'P'$ .  
 But  $OMP = OS'P$ ,  
 and  $OM'P' = OS'P'$ ,  
 $\therefore OS'P$  is the supplement of  $OS'P'$ .

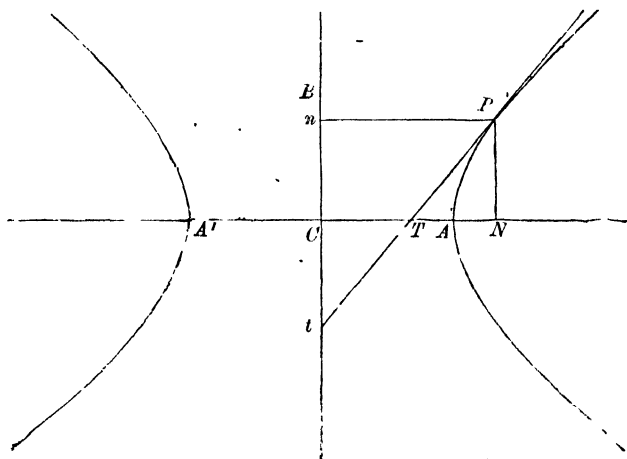
Hence the angles which  $OP$  and  $OP'$  subtend either at  $S$  or  $S'$  are supplementary.

In a similar manner if  $P$  and  $P'$  are on the same branch of the hyperbola, the angles subtended either at  $S$  or  $S'$  may be shown to be equal.

## PROP. XV.

55. If the tangent at any point  $P$  of an hyperbola meet the conjugate axis in the point  $t$ , and  $Pn$  be drawn at right angles to  $CB$ ; then

$$Cn \cdot Ct = BC^2.$$



Draw  $PN$  at right angles to  $CA$ ; then

$$Ct : CT :: PN : NT,$$

$$\therefore Ct : PN :: CT : NT,$$

$$\therefore Ct \cdot Cn : PN^2 :: CT \cdot CN : CN \cdot NT;$$

$$\text{or } Ct \cdot Cn : CT \cdot CN :: PN^2 : CN \cdot NT,$$

$$:: BC^2 : AC^2. \quad (\text{Prop. X.})$$

$$\text{But } CT \cdot CN = AC^2,$$

$$\therefore Ct \cdot Cn = BC^2.$$

56. The proofs that we have given up to this point of the properties of the hyperbola are closely analogous to the corresponding propositions in the ellipse. The remaining properties of the hyperbola are more conveniently investigated by means of its relation to certain lines, which we shall presently define, called *Asymptotes*, in the same manner as many of the properties of the ellipse were deduced from those of the auxiliary circle.

DEF. The hyperbola described (*see fig. Prop. XVI.*) with  $C$  as centre, and  $BB'$  as transverse axis, and  $AA'$  as conjugate axis, is called the *Conjugate Hyperbola*. Its foci, which will be on the line  $BCB'$ , will evidently be at the same distance from  $C$  as those of the original hyperbola, since

$$CS^2 = CA^2 + CB^2.$$

### PROP. XVI.

If through any point  $R$  on either of the diagonals of the rectangle formed by drawing tangents to the hyperbola and its conjugate at the vertices,  $A, A', B, B'$ , two ordinates  $RPN, RDM$ , be drawn at right angles to  $AA'$  and  $BB'$ , and meeting either the hyperbola or its conjugate in the points  $P$  and  $D$ ; then

$$RN^2 \cup PN^2 = BC^2,$$

$$\text{and } RM^2 \cup DM^2 = AC^2.$$

Let  $R$  be a point on the diagonal  $O'CO$ ; then

$$RN^2 : CN^2 :: AO^2 : AC^2,$$

$$:: BC^2 : AC^2;$$

$$\text{and } PN^2 : CN^2 - CA^2 :: BC^2 : AC^2; \text{ (Prop. X.)}$$

$$\therefore RN^2 - PN^2 : CA^2 :: BC^2 : AC^2.$$

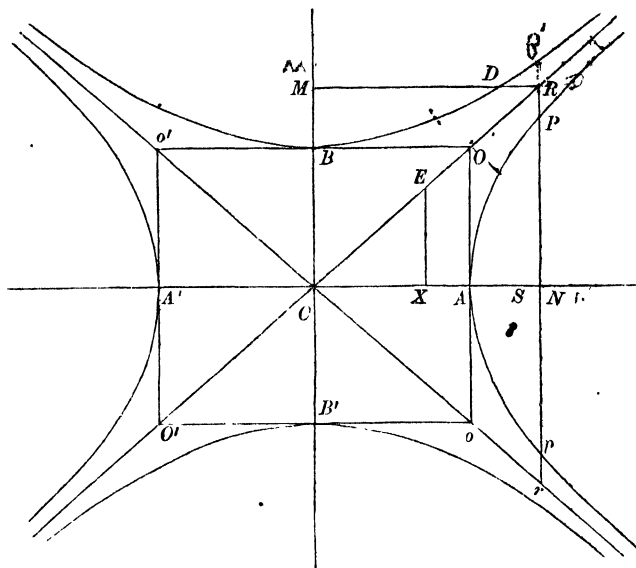
$$\therefore RN^2 - PN^2 = BC^2.$$

$$\text{Again, } RM^2 : CM^2 :: AC^2 : BC^2,$$

$$\text{and } DM^2 : CM^2 - CB^2 :: AC^2 : BC^2; \text{ (Prop. X.)}$$

$$\therefore RM^2 - DM^2 : BC^2 :: AC^2 : BC^2$$

$$\therefore RM^2 - DM^2 = AC^2.$$



In exactly the same manner, if  $NR$  had been produced to meet the conjugate hyperbola in  $P$ , and  $MR$  had been produced to meet the original hyperbola in  $D$ , we should have had,

$$PN^2 - RN^2 = BC^2,$$

$$\text{and } DM^2 - RM^2 = AC^2.$$

COR. If  $RP$  be produced to meet the hyperbola in  $p$ , and the other asymptote in  $r$ ; then

$$RN^2 - PN^2 = RP \cdot Pr; \text{ (Euclid, II. 5.)}$$

$$\therefore RP \cdot Pr = BC^2.$$

Hence as  $RPN$  is further removed from  $A$ , and the line  $Pr$  consequently increases, since the rectangle contained by  $RP$  and  $Pr$  remains constant,  $RP$  must diminish, and by taking  $R$  sufficiently far from  $C$ ,  $RP$  may be made less than any assignable magnitude. The line  $CR$ , therefore, continually approaches nearer and nearer to the hyperbola, though it never actually reaches it.

On account of this property,  $CR$  is called an *Asymptote* to the hyperbola.

So also if  $P$  be the point where  $NR$  produced meets the conjugate hyperbola, we shall have

$$RP \cdot Pr = BC^2;$$

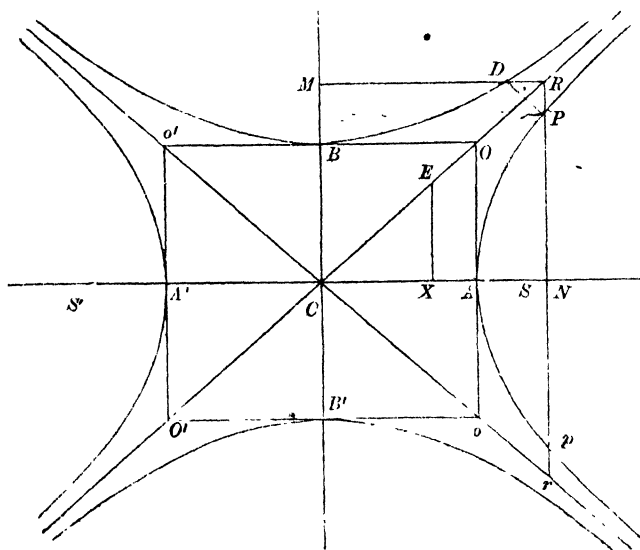
and therefore  $CR$  is also the asymptote to the conjugate hyperbola.

In the same manner it may be shown that the other diameter  $oCo'$  of the rectangle  $OO'$  is an asymptote to both hyperbolas. ¶

### PROP. XVII.

57. If  $E$  be the point where the asymptote meets the directrix, then

$$CE = AC.$$



For by similar triangles,

$$CE : CO :: CX : CA, \\ :: CA : CS. \text{ (Prop. II.)}$$

$$\text{But } CO^2 = CA^2 + CB^2 = CS^2; \checkmark$$

$$\therefore CO = CS;$$

$$\therefore CE = AC.$$

COR. If  $SE$  be joined, since

$$CE^2 = CA^2 = CS \cdot CX,$$

$\therefore$  the angle  $CES$  is a right angle. (*Euclid*, VI. 8, *Cor.*)

### PROP. XVIII.

If from any point  $R$  in one of the asymptotes to an hyperbola ordinates  $RPN$ ,  $RDM$  be drawn to the hyperbola and its conjugate respectively, and  $PD$  be joined,  $PD$  will be parallel to the other asymptote.

$$\text{For } RN^2 : RM^2 :: BC^2 : AC^2;$$

$$\text{and } RN^2 - PN^2 : RM^2 - DM^2 :: BC^2 : AC^2, \text{ (Prop. XVI.)}$$

$$\therefore PN^2 : DM^2 :: BC^2 : AC^2;$$

$$:: RN^2 : RM^2,$$

$$\therefore PN : DM :: RN : RM;$$

$$\therefore PD \text{ is parallel to } MN. \text{ (Euclid, VI. 2.)}$$

$$\text{Also } CN : CM :: AC : BC,$$

$$\therefore MN \text{ is parallel to } AB;$$

$$\text{and } OA : Ao :: OB : Bo',$$

$$\therefore AB \text{ is parallel to } oo'.$$

$$\text{Hence } PD \text{ is parallel to } oo'.$$

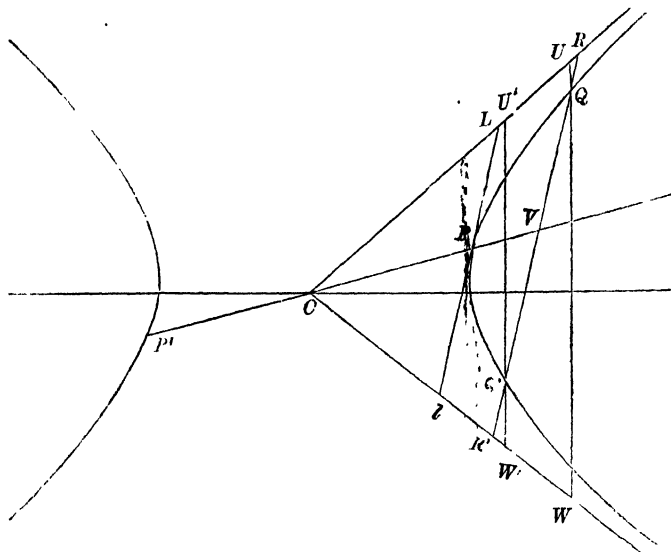
COR. So also if  $R$  and  $D$  be the points where  $NR$  and  $MR$  produced meet respectively the conjugate and the original hyperbola,  $PD$  will be still parallel to  $oo'$ .



## PROP. XIX.

58. If through any two points  $Q$  and  $Q'$  of an hyperbola a line  $RQQR'$  be drawn in any direction meeting the asymptotes in  $R$  and  $R'$ ; then will

$$RQ = R'Q'.$$



Through  $Q$  and  $Q'$  draw the ordinates  $UQW$ ,  $U'Q'W'$ ; meeting the asymptotes in  $U$ ,  $W$ ,  $U'$ ,  $W'$ ; then by similar triangles,

$$QR : QU :: Q'R : Q'U',$$

$$\text{and } QR' : QW :: Q'R' : Q'W';$$

$\therefore$  compounding

$$QR \cdot QR' : QU \cdot QW :: Q'R \cdot Q'R' : Q'U' \cdot Q'W'.$$

But  $QU \cdot QW = BC^2 = Q'U' \cdot Q'W'$ , (*Prop. XVI. Cor.*)

$$\therefore QR \cdot QR' = Q'R \cdot Q'R';$$

$$\begin{aligned}
 &\text{but } QR \cdot QR' = QR \cdot QQ' + QR \cdot Q'R', \\
 &\text{and } Q'R \cdot Q'R' = Q'R' \cdot QQ' + QR \cdot Q'R'; \\
 &\therefore QR \cdot QQ' = Q'R' \cdot QQ', \\
 &\therefore QR = Q'R'.
 \end{aligned}$$

COR. 1. If  $RQQ'R'$  move parallel to itself until the points  $Q$  and  $Q'$  coincide, the line  $RQR'$  will ultimately assume the position  $LP l$ , and will become a tangent to the hyperbola at  $P$ .

Hence, since  $RQ$  is always equal to  $R'Q'$ ,

$$LP = Pl,$$

or the tangent  $LP l$  is bisected at the point of contact  $P$ .

COR. 2. If  $CP$  be produced to meet  $RR'$  in  $V$ , then since

$$RV : VR' :: LP : Pl,$$

$$\therefore RV = VR';$$

$$\text{and } RQ = Q'R',$$

$$\therefore QV = Q'V.$$

Hence if a series of parallel chords be drawn in an hyperbola, their middle points will all be in the line drawn through the centre and the point where the tangent parallel to the chords meets the hyperbola.

DEF. A line  $PCP'$  drawn through the centre, and meeting the hyperbola in  $P$  and  $P'$ , is called a *Diameter*.

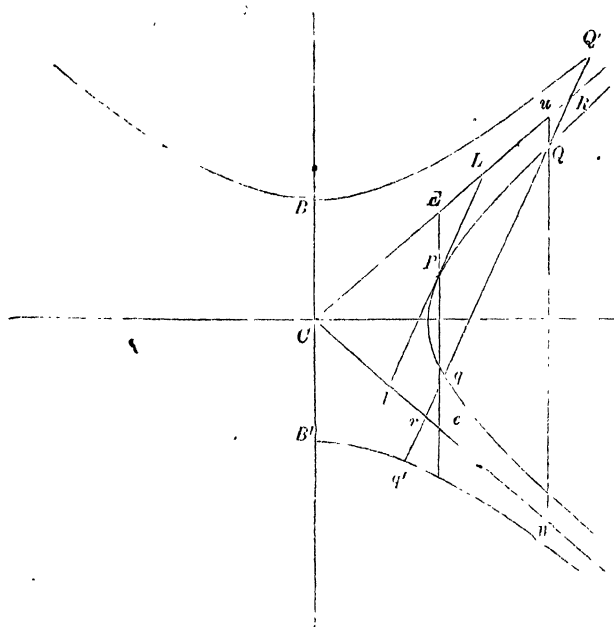
A diameter consequently bisects all chords drawn parallel to the tangents at its extremities.

## PROP. XX.

59. If through any point  $Q$  of an hyperbola a line  $RQr$  be drawn in any direction meeting the asymptotes in  $R$  and  $r$ , and  $LP l$  be the tangent drawn parallel to  $RQr$ ; then

$$RQ \cdot Qr = PL^2.$$

Through  $P$  and  $Q$  draw the ordinates  $EPe$ ,  $UQW$ , meeting the asymptote in  $E$ ,  $e$ ,  $U$ ,  $W$ ; then by similar triangles,



$$QR : QU :: PL : PE,$$

$$Qr : QW :: Pl : Pe,$$

$$\therefore QR \cdot Qr : QU \cdot QW :: PL \cdot Pl : PE \cdot Pe$$

but  $QU \cdot QW = BC^2 = PE \cdot Pe$ . (*Prop. XVI. Cor.*)

$$\therefore QR \cdot Qr = PL \cdot Pl,$$

$$= PL^2. \text{ (*Prop. XIX. Cor. 1.*)}$$

COR. If  $Qq$  be produced to meet the conjugate hyperbola in  $Q', q'$ , we may show that

$$Q'R \cdot Q'r = PL^2,$$

and also, as in Proposition XIX., that

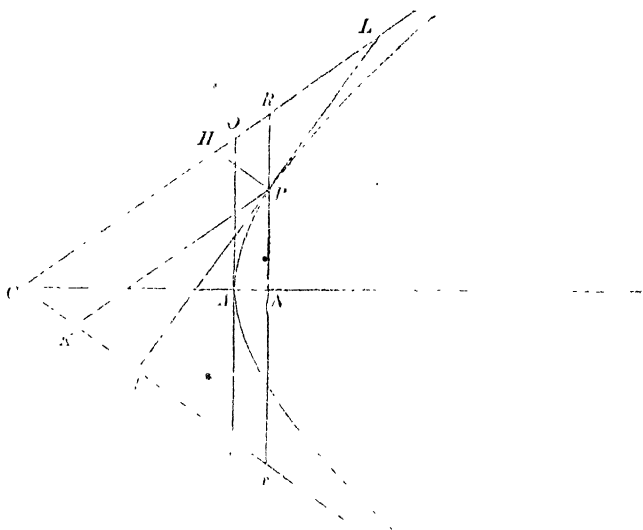
$$Q'R = q'r,$$

$$\therefore QQ' = qq'.$$

Hence if a line be drawn in any direction meeting both the hyperbolas, the portions intercepted between the hyperbola and its conjugate will be equal.

## PROP. XXI.

60, If from any point  $P$  of an hyperbola,  $PH$  and  $PK$  be drawn parallel to the asymptotes, meeting them in  $H$  and  $K$  respectively; then  $4 \cdot PH \cdot PK = CS^2$ .



Draw the ordinate  $RPNr$  meeting the asymptotes in  $H$  and  $r$ ; then by similar triangles,

$$PH : PR :: CO : Co,$$

$$\text{and } PK : Pr :: CO : Co,$$

$$\therefore PH \cdot PK : PR \cdot Pr :: CO^2 : Co^2,$$

$$:: CS^2 : 4BC^2.$$

$$\text{But } PR \cdot Pr = BC^2,$$

$$\therefore 4 \cdot PH \cdot PK = CS^2.$$

## PROP. XXII.

If the tangent at any point  $P$  of an hyperbola meet the asymptotes in  $L$  and  $l$ ; then the area of the triangle  $LCl$  is equal to the rectangle contained by  $AC$  and  $BC$ .

Draw  $PH$  and  $PK$  parallel to the asymptotes meeting them in  $H$  and  $K$ ; then

$$\text{since } CL : CH :: Ll : Pl,$$

$$\text{and } Ll = 2 Pl, \quad (\text{Prop. XIX. Cor. 1.})$$

$$\therefore CL = 2 CH = 2 PK;$$

$$\text{so } Cl = 2 CK = 2 PH,$$

$$\therefore CL \cdot Cl = 4 PH \cdot PK = CS^2, \quad (\text{Prop. XXI.})$$

$$= CO \cdot Co,$$

$$\therefore CL : CO :: Co : Cl,$$

$\therefore$  the triangles  $LCl$ ,  $OCo$  have the angle at  $C$  common and the sides about those angles reciprocally proportional;

$$\therefore \text{the triangle } LCl = \text{the triangle } OCo,$$

$$= AC \cdot AO,$$

$$= AC \cdot BC.$$

## PROP. XXIII.

61. If from any point  $R$  in the asymptote of an hyperbola, two ordinates  $RPN$  and  $RDM$  be drawn to the hyperbola and its conjugate respectively, then the tangents at  $P$  and  $D$  will be parallel respectively to  $CD$  and  $CP$ .

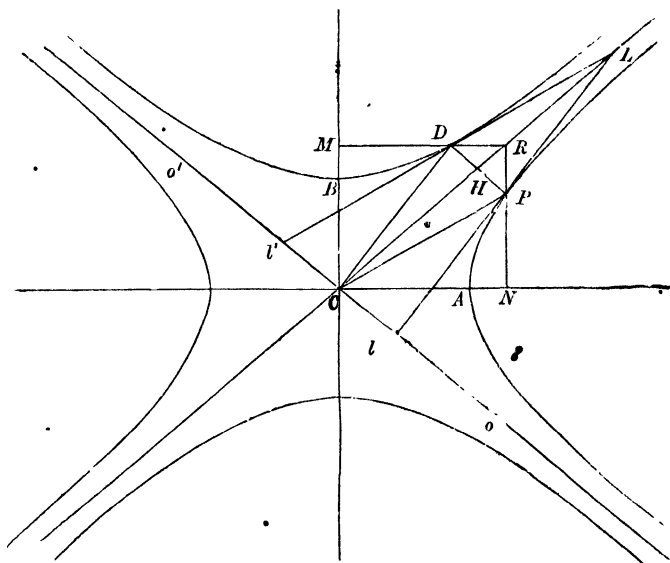
Join  $PD$ , meeting  $CR$  in  $H$ ; then

$$\text{since } PD \text{ is parallel to } oo', \quad (\text{Prop. XVIII.})$$

the tangents at  $P$  and  $D$  will each meet  $CR$  produced in the same point  $L$ . (*Prop. XXII.*)

Produce  $LP$  and  $LD$  to meet the other asymptotes in  $l$  and  $l'$ ; then

$$\text{since } CL \cdot Cl = CS^2 = CL' \cdot Cl', \quad (\text{Prop. XXII.})$$



$$\therefore Cl = Cl',$$

$$\therefore lC : Cl' :: lP : PL,$$

$\therefore CP$  is parallel to the tangent at  $D$ .

$$\text{Also } l'D : DL :: l'C : Cl,$$

$\therefore CD$  is parallel to the tangent at  $P$ .

The lines  $CP$  and  $CD$  are called *Conjugate Diameters*, since each of these lines is parallel to the tangent at the extremity of the other.

#### PROP. XXIV.

If  $CP$  and  $CD$  be semi-conjugate diameters in the hyperbola; then

$$CP^2 \hookrightarrow CD^2 = CA^2 \hookrightarrow CB^2.$$

Draw the ordinates  $NPR$ ,  $MDR$  meeting the asymptote in the point  $R$  (*Prop. XXIII.*); then

H

$$CR^2 - CP^2 = NR^2 - NP^2, \\ = BC^2, \text{ (Prop. XVI.)}$$

$$\therefore CR^2 = CP^2 + BC^2;$$

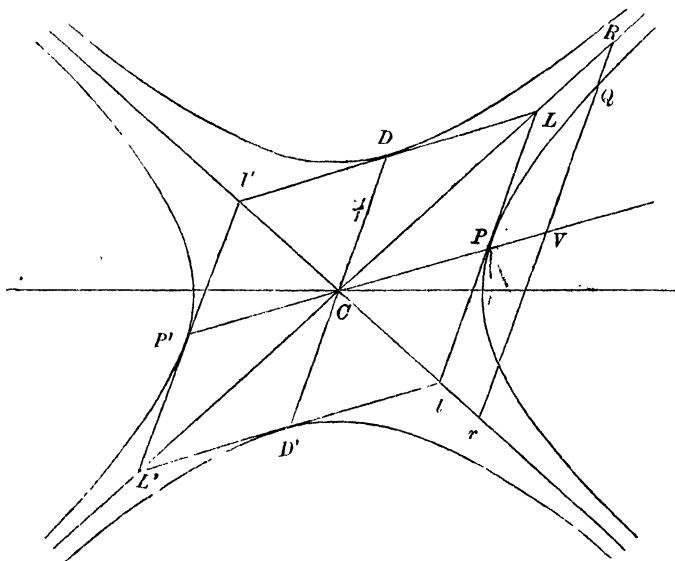
$$\text{so } CR^2 = CD^2 + AC^2,$$

$$\therefore CP^2 + BC^2 = CD^2 + AC^2;$$

$$\text{or } CP^2 \hookrightarrow CD^2 = AC^2 \hookrightarrow BC^2.$$

## PROP. XXV.

62. The area of any parallelogram formed by drawing tangents to the hyperbola and its conjugate at the extremities  $P, P', D, D'$  of a pair of conjugate diameters is equal to the rectangle contained by the axes.



Let  $LLL'L'$  be the parallelogram formed by drawing tangents at the extremities  $P, P', D, D'$ , of any pair of conjugate diameters. The points  $L, L', l, l'$ , will (Prop. XXIII.) be on the asymptotes.

Now the parallelogram  $LL' = 4$  parallelogram  $CL$ ,  
 $= 4$  triangle  $LCl$ ,  
 $= 4AC \cdot BC$ , (*Prop. XXII.*)  
 $= AA' \cdot BB'$ .

COR. If  $PF$  be drawn perpendicular to  $CD$ , then  
 $PF \cdot CD = AC \cdot BC$ .

Also, if the normal  $PG$  meet the transverse axis in  $G$ , as in the ellipse

$$PF \cdot PG = BC^2.$$

63. DEF. The line  $QV$  drawn from any point  $Q$  of the hyperbola parallel to the tangent at any point  $P$ , and meeting  $CP$  produced in  $V$ , is called an *Ordinate* to the diameter  $CP$ .

PROP. XXVI.

If  $QV$  be an ordinate to the diameter  $P'CP$ , and  $CD$  be conjugate to  $CP$ ; then

$$QV^2 : PV \cdot P'V :: CD^2 : CP^2.$$

Produce  $VQ$  to meet the asymptotes in  $R$  and  $r$ ; and let the tangent at  $P$  meet the asymptotes in  $L$  and  $l$ ; then

$$RV^2 : PL^2 :: CV^2 : CP^2,$$

$$\therefore RV^2 - PL^2 : PL^2 :: CV^2 - CP^2 : CP^2.$$

But  $RQ \cdot Qr = PL^2$ , (*Prop. XX.*)

$$\therefore RV^2 - QV^2 = PL^2,$$

$$\text{or } RV^2 - PL^2 = QV^2.$$

And  $CV^2 - CP^2 = PV \cdot P'V$ , (*Euclid, II. 6*)

$$\therefore QV^2 : PL^2 :: PV \cdot P'V : CP^2.$$

Alternately,  $QV^2 : PV \cdot P'V :: PL^2 : CP^2$ .

But since  $PD$  is a parallelogram, (*Prop. XXIII.*)

$$\therefore PL = CD.$$

Hence  $QV^2 : PV \cdot P'V :: CD^2 : CP^2$ .

COR. If  $VQ$  be produced to meet the conjugate hyperbola in  $Q'$ , then

since  $Q'R \cdot Q'r = PL^2$ , (*Prop. XX. Cor.*)

$$\therefore Q'V^2 - RV^2 = PL^2.$$

Hence  $Q'V^2 : CV^2 + CP^2 :: CD^2 : CP^2$

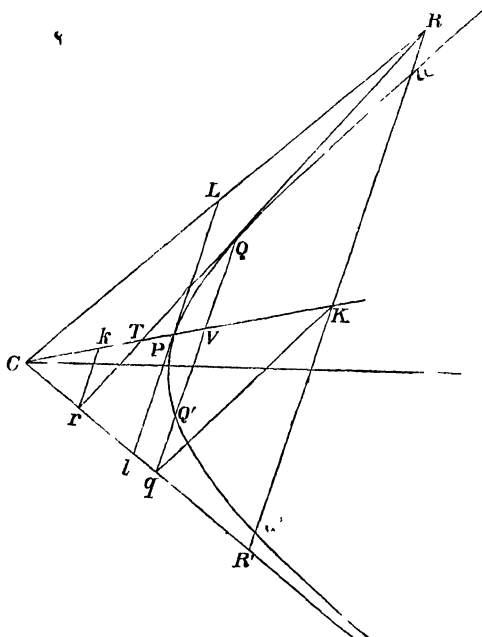


## PROP. XXVII.

64. If  $QV$  be an ordinate to the diameter  $PV$ , and the tangent at  $Q$  meet  $CP$  in the point  $T$ ; then

$$CV \cdot CT = CP^2.$$

Draw the tangent  $LPl$  meeting the asymptotes in the points  $L, l$ ; also let the tangent at  $Q$  meet the asymptotes in  $R, r$ .



Draw  $RK, rk$  parallel to  $QV$  meeting  $CP$  in  $K, k$ .

Now since the triangles  $RCr, LCl$  are equal, (*Prop. XXII.*) and have the angle at  $C$  common,

$$\therefore CR : CL :: Cl : Cr. \text{ (Euclid, VI. 15.)}$$

$$\text{But } CR : CL :: CK : CP,$$

$$\begin{aligned}
 &\text{and } Cl : Cr :: CP : Ck, \\
 &\therefore CK : CP :: CP : Ck, \\
 &\therefore CK \cdot Ck = CP^2.
 \end{aligned}$$

Again, produce  $RK$  and  $QV$  to meet the asymptote  $Cl$  in  $R'$  and  $q$ ; then

Since  $Rr$  is bisected in  $Q$ , (*Prop. XIX. Cor. 1*)

$\therefore R'r$  is bisected in  $q$ ,

and  $RK = R'K$ , (*Prop. XIX. Cor. 2*)

$\therefore Kq$  is parallel to  $Rr$ ,

$\therefore CT : CK :: Cr : Cq$ ,

$:: Ck : CV$ ,

$$\begin{aligned}
 \therefore CV \cdot CT &= CK \cdot Ck \\
 &= CP^2.
 \end{aligned}$$

COR. 1. Conversely, if  $QV$  be an ordinate to  $PV$ ,

and  $CV \cdot CT = CP^2$ ,

then  $QT$  is the tangent at  $Q$ .

COR. 2. Hence also, if  $RR'$  meet the curve in  $U$  and  $U'$ ,

and  $kU, kU'$  be drawn,

since  $CK \cdot Ck = CP^2$ ,

$\therefore kU$  and  $kU'$  are tangents to the hyperbola at  $U$  and  $U'$ .

### PROP. XXVIII.

65. If two chords of a hyperbola intersect one another, the rectangles contained by their segments are proportional to the squares of the diameters parallel to them.

Let  $QOq$  be any chord drawn through the point  $O$ , and let  $CD$  be drawn parallel to it, meeting the conjugate hyperbola in  $D$ .

Produce  $Qq$  to meet the asymptotes in  $R$  and  $r$ ; and draw the diameter  $CPV$ , bisecting both  $Qq$  and  $Rr$  in  $V$ . (*Prop. XIX. Cor. 2*.)

Also draw the tangent  $LPl$  parallel to  $Qq$ , meeting the asymptotes in  $L$  and  $l$ .



but  $P\dot{U} . PW = BC^2$ , (*Prop. XVI.*)  
 and  $PL^2 = CD^2$ , (*Prop. XXIII.*)

$$\therefore RO . rO : OE . Oe :: CD^2 : BC^2,$$

$$\text{or } \dot{R}O . rO : CD^2 :: OE . Oe : BC^2,$$

$$\therefore RO . rO - PL^2 : CD^2 :: OE . Oe - BC^2 : BC^2,$$

$$\text{or } QO . Oq : CD^2 :: OE . Oe - BC^2 : BC^2.$$

In the same manner if through  $O$  another chord  $Q'Oq'$  be drawn, and  $CD'$  be drawn parallel to it, meeting the conjugate hyperbola in  $D'$ , we shall have

$$Q'O . Oq' : CD'^2 :: OE . Oe - BC'^2 : BC'^2.$$

$$\text{Hence } QO . Oq : Q'O . Oq' :: CD^2 : CD'^2.$$

COR. The same result may be shown to be true when the point  $O$  is outside the hyperbola. Moreover, it is not necessary that the chords should be drawn meeting *one* branch only of the hyperbola or the *same* branch. The proportion still holds good when one or both of the chords meet both branches of the hyperbola, or when the chords are drawn in different branches.

66. DEF. If with a point  $O$  on the normal at  $P$  as centre, and  $OP$  as radius, a circle be described touching the hyperbola at  $P$ , and cutting it in  $Q$ ; then when the point  $Q$  is made to approach indefinitely near to  $P$ , the circle is called the *Circle of Curvature* at the point  $P$ .

### PROP. XXIX.

If  $PH$  be the chord of the circle of curvature at the point  $P$  of a hyperbola, which passes through the centre; then

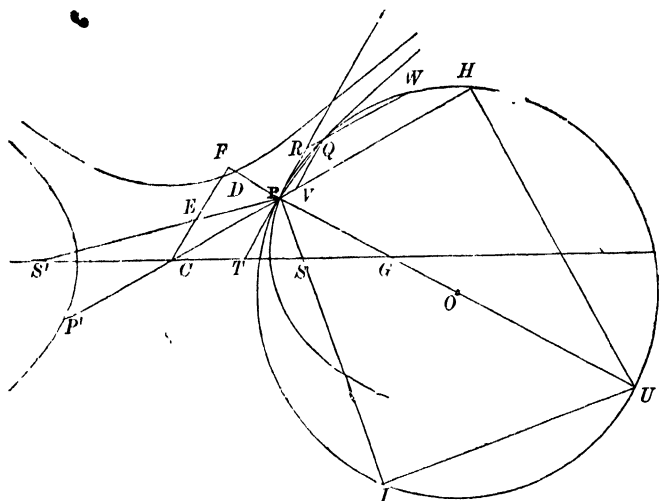
$$PH . CP = 2CD^2.$$

Let  $PT$  be the tangent, and  $PG$  the normal at the point  $P$ .

With centre  $O$ , and radius  $OP$ , describe a circle cutting the hyperbola in the point  $Q$ .

Draw  $RQW$  parallel to  $CP$ , meeting the circle in  $W$ , and  $TP$  produced in  $R$ .

Also, draw  $QV$  parallel to  $PR$ , meeting the diameter  $PP'$  in  $V$ ; then since  $RP$  touches the circle at  $P$ ,



$$\therefore RQ \cdot RW = PR^2, \text{ (Euclid, III. 36)}$$

$$\text{or } PV \cdot RW = QV^2.$$

$$\text{But } QV^2 : PV \cdot P'V :: CD^2 : CP^2, \text{ (Prop. XXVI.)}$$

$$\therefore PV \cdot RW : PV \cdot P'V :: CD^2 : CP^2,$$

$$\text{or } RW : P'V :: CD^2 : CP^2,$$

Now, when the circle becomes the circle of curvature at  $P$ , the points  $R$  and  $Q$  move up to, and coincide with  $P$ , and the lines  $RW$  and  $PH$  become equal, while

$$P'V \text{ becomes equal to } PP', \text{ or } 2CP.$$

$$\text{Hence, } PH : 2CP :: CD^2 : CP^2,$$

$$\therefore PH \cdot CP : 2CP^2 :: 2CD^2 : 2CP^2,$$

$$\therefore PH \cdot CP = 2CD^2.$$

### PROP. XXX.

If  $PU$  be the diameter of the circle of curvature at the point  $P$  of the hyperbola, and  $PF$  be drawn at right angles to  $CD$ ; then

$$PU \cdot PF = 2CD^2.$$

Since the triangle  $PHU$  is similar to the triangle  $PFC$ ,

$$\therefore PU : PH :: CP : PF,$$

$$\therefore PU \cdot PF = PH \cdot CP, \\ = 2 CD^2. \quad (\text{Prop. XXIX.})$$

### PROP. XXXI.

If  $PI$  be the chord of the circle of curvature through the focus of the hyperbola; then

$$PI \cdot AC = 2 CD^2.$$

Let  $S'P$  meet  $CD$  in  $E$ ; then since the triangles  $PIU$  and  $PEF$  are similar,

$$\therefore PI : PU :: PF : PE.$$

$$\text{But } PE = AC, \quad (\text{Prop. XII. Cor.})$$

$$\therefore PI : PU :: PF : AC,$$

$$\therefore PI \cdot AC = PU \cdot PF, \\ = 2 CD^2. \quad (\text{Prop. XXX.})$$

The point where the circle of curvature intersects the hyperbola may be determined as in the case of the ellipse.



### PROP. XXXII.

67. If  $P$  be any point on the hyperbola, and  $CD$  be conjugate to  $CP$ ; then

$$SP \cdot S'P = CD^2.$$

Draw  $PII'$  parallel to the asymptote  $CE$  meeting the directrices in  $I$  and  $I'$ , and  $CB'$  in  $U$ .

Let the ordinates  $NP$ ,  $MD$  meet the asymptote in  $R$ , and draw  $PW$  perpendicular to the directrix; then by similar triangles,

$$PI : PW :: CE : CX, \\ :: CA : CX. \quad (\text{Prop. XVII.})$$



## PROBLEMS ON THE HYPERBOLA.

1. THE locus of the centre of a circle touching two given circles is a hyperbola or ellipse.

2. If on the portion of any tangent intercepted between the tangents at the vertices a circle be described, it will pass through the foci.

3. In a hyperbola the tangents at the vertices will meet the asymptotes in the circumference of the circle described on  $SS'$  as diameter.

4. If from a point  $P$  in a hyperbola  $PII'$  be drawn parallel to the transverse axis meeting the asymptotes in  $I$  and  $I'$ ; then  $PI \cdot PI' = AC^2$ .

5. If a circle be inscribed in the triangle  $SPS'$ , the locus of its centre is the tangent at the vertex.

6. If  $PN$  be the ordinate of the point  $P$ , and  $NQ$  a tangent to the circle described on the transverse axis as diameter, and  $PM$  be drawn parallel to  $QC$  meeting the axis in  $M$ , then  $MN = BC$ .

7. If  $PN$  be the ordinate of a point  $P$ , and  $NQ$  be drawn parallel to  $AP$  to meet  $CP$  in  $Q$ , then  $AQ$  is parallel to the tangent at  $P$ .

8. If a hyperbola and an ellipse have the same foci, they cut one another at right angles.

9. If the tangent at  $P$  intersect the tangents at the vertices in  $R, r$ , and the tangent at  $P'$  intersect them in  $R', r'$ , then  $AR \cdot Ar = AR' \cdot Ar'$ .



10. If any two tangents be drawn to a hyperbola, the lines joining the points where they intersect the asymptotes will be parallel.

11. The perpendicular drawn from the focus to the asymptotes of a hyperbola is equal to the semi-conjugate axis.

12. If the asymptotes meet the tangent at the vertex in  $O$ , and the directrix in  $E$ ; then  $AE$  is parallel to  $SO$ .

13. In a rectangular hyperbola conjugate diameters are equal to one another.

14. In a rectangular hyperbola the normal  $PG$  is equal to  $CP$ .

15. The lines drawn from any point in a rectangular hyperbola to the extremities of a diameter make equal angles with the asymptotes.

16. Prove that the asymptotes to a hyperbola bisect the lines joining the extremities of conjugate diameters.

17. A line drawn through one of the vertices of a hyperbola and terminated by two lines drawn through the other vertex parallel to the asymptotes will be bisected at the other point where it cuts the hyperbola.

18.  $P$  is any point on a hyperbola, and  $P'$  a point on the conjugate hyperbola. If  $CP$  and  $CP'$  be conjugate, prove that

$$S'P' - SP = AC - BC,$$

$S$  and  $S'$  being the interior foci.

19. If  $CP$  and  $CD$  be conjugate, and through  $C$  a line be drawn parallel to either focal distance of  $P$ , the perpendicular from  $D$  upon this line is equal to  $BC$ .

20. Given a pair of conjugate diameters, find the principal axes.

21. If  $Q$  be a point on the conjugate axis of a rectangular hyperbola, and  $QP$  be drawn parallel to the transverse axis meeting the curve in  $P$ ; then

$$PQ = AQ.$$

22. In a rectangular hyperbola the focal chords drawn parallel to conjugate diameters are equal.

23. If in an equilateral hyperbola  $CY$  be drawn at right angles to the tangent at  $P$ , and  $AY$  be joined, the triangles  $PCA$ ,  $CAY$  are similar.

24. The radius of the circle which touches a hyperbola and its asymptotes, is equal to that part of the latus rectum produced which is intercepted between the curve and the asymptotes.

25. If  $QQ'$  be any chord of a hyperbola, and  $CP$  the diameter corresponding to it, and  $QH$ ,  $PK$ ,  $Q'H'$  be drawn parallel to one asymptote meeting the other in  $H$ ,  $K$  and  $H'$ , then  $CH \cdot CH' = CK^2$ .

26. If the chord  $RPP'R'$  intersect the hyperbola in the points  $P$ ,  $P'$ , and the asymptotes in  $R$ ,  $R'$ ; and  $PK$  be drawn parallel to  $CR'$ , and  $P'K'$  to  $CR$ ; then  $RK = P'K'$ , and  $R'K' = PK$ .

27. If  $AA'$  be any diameter of a circle, and  $PNQ$  an ordinate to it, then the locus of the intersections of  $AP$ ,  $A'Q$  is a rectangular hyperbola.

28. If two concentric rectangular hyperbolas be described, the axes of one being the asymptotes of the other, they will intersect at right angles.

29. If any chord  $AP$  through the vertex be divided in  $Q$ , so that  $AQ : QP :: AC^2 : BC^2$ , and  $QN$  be drawn to the foot of the ordinate  $PN$ , prove that a straight line drawn at right angles to  $QN$  from  $Q$  cuts the transverse axis in the same ratio.

30. Prove that the curve which trisects the arcs of all segments of a circle described upon a given base is a hyperbola.

31. If  $SVs$ ,  $TVt$  be two tangents cutting one asymptote in  $S$ ,  $T$ , and the other in  $s$ ,  $t$ , prove that

$$VS : Vs : Vt : VT.$$

32. If from the exterior focus of a hyperbola a circle be described with radius equal to  $BC$ , and tangents be drawn to it from any point in the hyperbola, the line joining the points of contact will touch the circle described on the transverse axis as diameter.

33. Circles are drawn touching the straight line  $AB$  in a fixed point  $C$ ; and from the fixed points  $A, B$  tangents are drawn to these circles. The locus of their intersection is an ellipse or hyperbola. Distinguish between the two cases.

34.  $PP'$  is a double ordinate in an ellipse.  $AP, A'P'$  are produced to meet in  $Q$ . Prove that the locus of  $Q$  is a hyperbola with the same axes as the ellipse.

35. If the tangent at  $P$  intersect the asymptotes in  $L$  and  $l$ , and  $PG$  be the normal at  $P$ , then the angle  $LGl$  is a right angle.

36. If an ellipse, a parabola, and a hyperbola, have a common tangent, and the same curvature at the vertex, the ellipse will be entirely within the parabola, and the parabola entirely within the hyperbola.

37. The chord  $RPP'R'$  of a hyperbola intersects the asymptotes in  $R$  and  $R'$ . From the point  $R$  a tangent  $RQ$  is drawn meeting the hyperbola in  $Q$ . If  $PH, QK, P'H'$  be drawn parallel to one asymptote meeting the other in the points  $H, K, H'$ ; then  $PH + P'H' = 2 QK$ .

38. If through  $P, P'$  on a hyperbola lines be drawn parallel to the asymptotes forming a parallelogram, of which  $PP$  is one diagonal; the other diagonal will pass through the centre.

39. If  $P$  be the middle point of a line  $EF$  which moves so as to cut off a constant area from the corner of a rectangle, its locus is an equilateral hyperbola.

40.  $PM, PN$  are drawn parallel to the asymptotes  $CN, CM$ , and an ellipse is constructed having  $CN, CM$  for semi-conjugate diameters. If  $CP$  cut the ellipse in  $Q$ , the tangents at  $Q$  and  $P$  to the ellipse and hyperbola are parallel.

41. If a circle be described through any point  $P$  of a given hyperbola and the extremities  $A, A'$  of the transverse axis, and  $NP$  be produced to meet the circle in  $Q$ ; prove that  $Q$  traces out a hyperbola whose conjugate axis is a third proportional to the conjugate and transverse axes of the original hyperbola.

42. If lines be drawn from any point of a rectangular hyperbola to the extremities of a diameter, the difference between the angles which they make with the diameter will be equal to the angle which this diameter makes with its conjugate.

43. If between a rectangular hyperbola and its asymptotes any number of concentric elliptic quadrants be inscribed, the rectangle contained by their axes will be constant.

44. In the rectangular hyperbola if  $CP$  be produced to  $Q$  so that  $PQ = CP$ , and  $QO$  be drawn at right angles to  $CQ$  to intersect the normal in  $O$ ,  $O$  is the centre of curvature at  $P$ .

45. With two conjugate diameters of an ellipse as asymptotes a pair of conjugate hyperbolas are constructed; prove that if one hyperbola touch the ellipse the other will do so likewise; prove also that the diameters drawn through the points of contact are conjugate to each other.

46. If a pair of conjugate diameters of an ellipse when produced be asymptotes to a hyperbola, the points of the hyperbola at which a tangent to the hyperbola will also be a tangent to the ellipse, lie in an ellipse similar to the given one.

47. In the rectangular hyperbola the radius of curvature at  $P$  is to the radius of curvature at  $P'$  in the triplicate ratio of  $CP$  to  $CP'$ .

48.  $OP$ ,  $OQ$  are tangents to an ellipse at  $P$  and  $Q$ , and asymptotes to a hyperbola. Show that a pair of their common chords is parallel to  $PQ$ . One of these chords being  $RS$ , prove that if  $PR$  touches the hyperbola at  $P$ ,  $QS$  touches it at  $S$ ; also if  $PS$ ,  $QR$  meet in  $U$ ,  $OU$  bisects  $PQ$ .

49. The base of the triangle  $ABC$  remains fixed, while the vertex  $C$  moves in an equilateral hyperbola passing through  $A$  and  $B$ . If  $P$ ,  $Q$  be the points in which  $AC$ ,  $BC$  meet the circle described on  $AB$  as diameter, the intersection of  $AQ$ ,  $BP$  is on the other branch of the hyperbola.

## CHAPTER IV.

### THE SECTIONS OF THE CONE.

68. DEF. If two indefinite straight lines  $IOI'$ ,  $DOD'$ , intersect one another at a point  $O$ , and one of them  $IOI'$  remain fixed while the other  $DOD'$  revolves round it in such a manner that its inclination to  $IOI'$  is the same in all positions, the surface generated by  $DOD'$  will be a *Right Cone*.

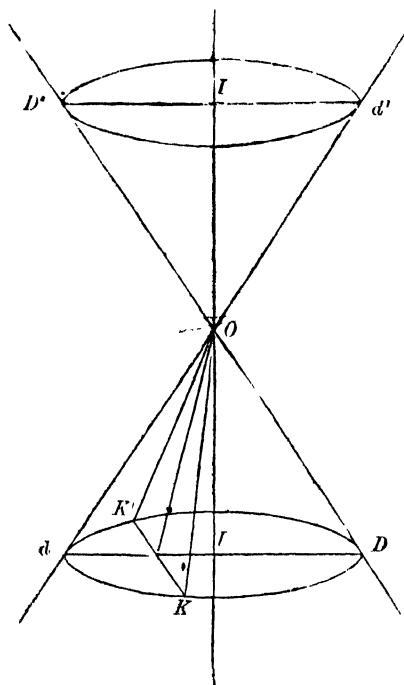
The line  $IOI'$  is called the *Axis*, and the point  $O$  the *Vertex* of the *Cone*.

It now remains for us to show (*see Introduction*) that the curve formed by the intersection of this surface with a plane is *in general* one of the three curves whose properties we have been investigating, and to consider under what circumstances it will be the Parabola, Ellipse, or Hyperbola.

If the cutting plane pass through the vertex of the cone as  $KOK'$ , and intersect the cone again *at all*, it will *in general* cut it in two straight lines as  $OK$ ,  $OK'$ , which will represent two positions of the generating line.

The inclination of these lines to each other will depend upon the inclination of the cutting plane to the axis of the cone, and will be greatest when this plane passes through the axis, in which case it will be double the constant angle between the axis and the generating line.

If the cutting plane pass through a generating line  $dod'$  and be perpendicular to the plane containing this line and the axis, it will simply touch the cone along this line.



Should the cutting plane not pass through the vertex, and be at right angles to the axis of the cone, the section will evidently be a circle.

In any other case the section will, as we proceed to show, be a Parabola, Ellipse, or Hyperbola.

Whatever be the position of the cutting plane with respect to the cone, we can always suppose a plane drawn through the axis of the cone at right angles to it; and it will be convenient to have this latter plane represented by the plane of the paper as  $DOd$ . The cutting plane will therefore always be taken at right angles to the plane  $DOd$  of the paper.

## CONIC SECTIONS.

## PROP. I.

69. The curve formed by the intersection of the surface of a right cone with a plane (which neither passes through its vertex nor is at right angles to its axis) will be a *Parabola*, *Ellipse*, or *Hyperbola*, according as the inclination of the cutting plane to the axis of the cone is *equal to*, *greater*, or *less than* the constant angle which the generating line forms with the axis.

Let the plane of the paper represent the plane drawn through the axis  $IOI'$  of the cone at right angles to the cutting plane; and let it intersect the surface of the cone in the two generating lines  $OD$ ,  $Od$ .

Let the cutting plane intersect the surface of the cone in the curve  $PA$ , and the plane of the paper in the line  $ANH$ .

The curve will evidently be symmetrical with respect to this line.

On  $AH$  take any point  $N$ , and through  $N$  draw a plane perpendicular to the axis meeting the surface of the cone in the circle  $RPr$ , and the cutting plane in the line  $PN$ , which will be at right angles to the plane of the paper and to  $AN$ .

Let a sphere be inscribed in the cone touching the cone in the circle  $EQe$  and the cutting plane in the point  $S$ , and let the plane  $EQe$  intersect the cutting plane in the line  $XM$ , which will be at right angles to the plane of the paper, and therefore parallel to  $PN$ .

Draw  $PM$  perpendicular to  $XM$ , and join  $PS$ ,  $PO$ , and let  $PO$  meet the circle  $EQe$  in the point  $Q$ .\*

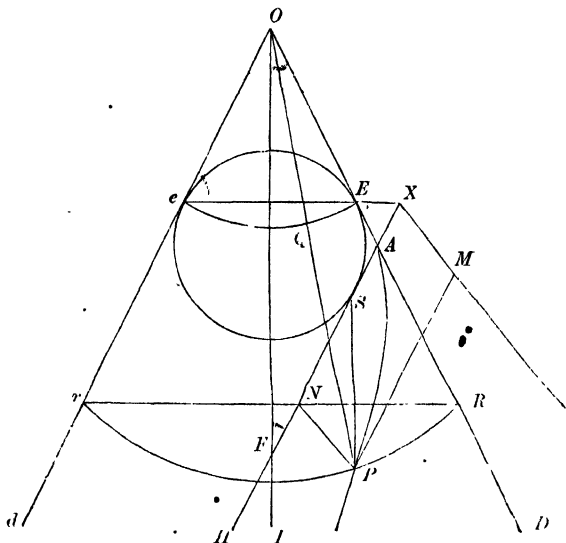
Then since  $PS$  and  $PQ$  are both tangents to the sphere,

$$\therefore PS = PQ.$$

$$\text{But } PQ = RE,$$

$$\therefore PS = RE.$$

\* N.B. In the figure, to avoid confusion, that part of the section which lies above the plane of the paper is alone represented.



But  $RE : XN :: AE : AX$ , (*Euclid*, VI. 2.)

and  $AE = AS$ ,

$\therefore RE : XN :: AS : AX$ ,

$\therefore SP : PM :: AS : AX$ ,

$\therefore$  the curve  $PA$  is either a *Parabola*, *Ellipse*, or *Hyperbola*, whose focus is  $S$  and directrix  $XM$ .

Again, let  $AH$  meet the axis  $OI$  in  $F$ .

Then the angle  $AFO$  will be the inclination of the cutting plane to the axis.

(1.) Let the angle  $AFO =$  the angle  $F Od$ ;

then  $AH$  is parallel to  $Od$ ,

$\therefore$  the angle  $AXE =$  the angle  $OeE =$  the angle  $AEX$ ,

$\therefore AE = AX$ ,

$\therefore AS = AX$ ,

$\therefore$  the curve  $AP$  will be a *Parabola*.





- (2.) Let the angle  $AFO$  be  $> FOd$ ; then  
 the complement  $FXE$  is  $<$  the complement  $O Ee$ ,  
 $\therefore$  the angle  $AXE$  is  $<$  the angle  $AEX$ ,  
 $\therefore AE$  is  $< AX$ ,  
 or  $AS$  is  $< AX$ ,  
 $\therefore$  the curve  $AP$  is an *Ellipse*.

Since the angles  $HFO$ ,  $FOd$  are together less than  $HFO$ ,  $OFA$ , i.e. than two right angles, the lines  $AH$  and  $Oe$  may be produced to meet in  $A'$ .

If another sphere be described touching the cone in the circle  $E'Q'e'$  and the cutting plane in the point  $S'$ ; and the line  $X'M'$  denote the intersection of the plane  $E'Q'e'$  with the cutting plane, and  $PM'$  be drawn at right angles to this line, it can easily be shown that

$$S'P : PM' :: S'A' : A'X'.$$

Hence  $S'$  and  $X'M'$  represent respectively the other focus and directrix of the ellipse.

Also if  $BC$  be the semi-axis minor, and through the centre  $C$  a line  $UCU'$  be drawn parallel to  $Ee$  meeting  $OD$ ,  $Od$  in  $U$  and  $U'$ , then it is evident that

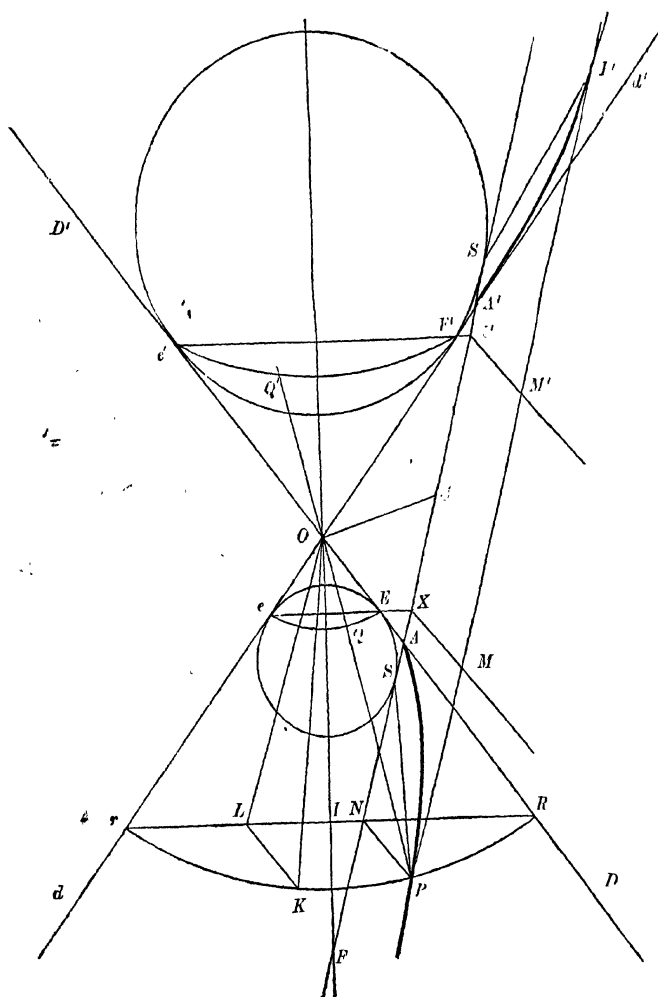
$$BC^2 = CU \cdot CU'.$$

- (3.) Let the angle  $AFO$  be  $< FOd$ ; then  
 the angle  $AXE$  is  $>$  the angle  $AEX$ ,  
 $\therefore AE$  is  $> AX$ ,  
 $\therefore AS$  is  $> AX$ ,  
 $\therefore$  the curve  $PA$  is an *Hyperbola*.

Since the angles  $AFO$ ,  $FOd'$  are less than the two  $FOd$ ,  $FOd'$ , i.e. than two right angles, the lines  $FA$  and  $dO$  may be produced to meet in  $A'$ .

In this case the cutting plane will intersect the other half of the cone, and if any point  $P'$  be taken on this part of the curve, and  $P'M$  be drawn at right angles to  $XM$ , it can be shown as before that

$$SP' : P'M :: SA : AX.$$



The intersection of the cutting plane therefore with this portion of the cone will be the other branch of the hyperbola.

Also if another sphere be described touching the upper portion of the cone in  $E'Q'e'$ , and the cutting plane in  $S'$ , and the line  $X'M'$  denote the intersection of the plane  $E'Q'e'$  with the cutting plane, and  $P'M'$  be drawn at right angles to this line, it can be easily shown that

$$S'P' : P'M' :: S'A' : A'X'.$$

Hence  $S'$  and  $X'M'$  will represent respectively the other focus and directrix of the hyperbola.

COR. 1. In this last case, *i.e.* when the section is an hyperbola, if a plane  $OKL$  be drawn through the vertex of the cone parallel to the cutting plane, meeting the plane of the paper in the straight line  $OL$ , and the surface of the cone in the generating line  $OK$ ; then

$$\begin{aligned} OL : OK &:: OL : OR, \\ &:: AN : AR, \\ &:: AX : AE, \text{ (Euclid, VI. 2)} \\ &:: AX : AS, \\ &:: CA : CS, \text{ (Chap. III. Prop. II.)} \end{aligned}$$

where  $C$  is the middle point of  $AA'$ , and therefore the centre of the hyperbola.

$\therefore KOL$  is half the angle between the asymptotes. (Chap. III. Prop. XVI.)

Again, if  $BC$  be the conjugate semi-axis, and  $CUU$  be drawn parallel to  $Rr$  meeting  $OD'$ ,  $Od'$  in  $U$  and  $U'$ , then

$$\begin{aligned} \text{since } CU : AC &:: RL : OL, \\ \text{and } CU' : A'C &:: rL : OL, \\ \therefore CU \cdot CU' : AC^2 &:: RL \cdot rL : OL^2, \\ &:: KL^2 : OL^2; \\ \text{but } BC^2 : AC^2 &:: KL^2 : OL^2, \\ \therefore BC^2 &= CU \cdot CU'. \end{aligned}$$

COR. 2. If the cutting plane is parallel to the axis,  $OL$  and  $OI$  coincide.



## PROBLEMS ON THE SECTIONS OF THE CONE.

1. THE foci of all parabolic sections which can be cut from a given right cone, lie upon the surface of another cone.

2. The foci of all elliptical sections of a given right cone, in which the ratio of  $CA$  to  $CS$  is the same, will lie on two other cones.

3. The extremities of the minor axes of the elliptical sections of a right cone made by parallel planes, lie on two generating lines.

4. The latus rectum of a parabola cut from a given cone varies as the distance between the vertices of the cone and the parabola.

5. Under what conditions is it possible to cut an equilateral hyperbola from a given right cone?

6. Two cones whose vertical angles are supplementary are joined as in *Art. 69, Cor. 2*. Prove that the latera recta of the curves of section of the cones, whose axes are respectively  $OI$  and  $Oi$ , made by planes parallel or perpendicular to the plane of the axes, are in the duplicate ratio of  $Oi : OI$ .

## ADDITIONAL PROBLEMS.

1. SHOW that the part of the directrix of a parabola, intercepted between the perpendiculars on it from the extremities of any focal chord, subtends a right angle at the focus.

2. The locus of the foci of all parabolas touching the three sides of a triangle is a circle. Prove this, and give a geometrical construction for finding the centre.

3. A system of parabolas which always touch two given straight lines have their axes parallel; show that the locus of the foci is a straight line.

4. Prove that the locus ~~at~~<sup>of</sup> the foot of the perpendicular from the focus of a parabola on the normal is a parabola.

5. If  $S$  be the focus of a parabola which touches the sides  $AB$ ,  $AC$  of the triangle  $ABC$  at the points  $B$ ,  $C$ , and  $O$  the centre of the circle described about the triangle; prove that the angle  $OSA$  is a right angle.

6. From the focus of a parabola a straight line is drawn parallel to the tangent at any point  $P$ , meeting the diameter through  $P$  in  $V'$ ; show that the tangent drawn from  $P$  to any circle passing through  $V'$  is equal to one-half of the ordinate  $QV$ ,  $V$  being the second point in which the circle cuts the diameter through  $P$ .

7.  $PSp$  is a focal chord, and upon  $PS$  and  $pS$  as diameters, circles are described; prove that the length of either of their common tangents is a mean proportional between  $AS$  and  $Pp$ .

8. If  $AQ$  be a chord of a parabola through the vertex  $A$ , and  $QR$  be drawn perpendicular to  $AQ$  to meet the axis in  $R$ ; prove that  $AR$  will be equal to the chord through the focus parallel to  $AQ$ .

9. The locus of the vertices of all parabolas, which have a common focus and a common tangent, is a circle.

10. Two parabolas have a common axis and vertex, and their concavities turned in opposite directions; the latus rectum of one is eight times that of the other; prove that the portion of a tangent to the former, intercepted between the common tangent and axis, is bisected by the latter.

11.  $B$  is a point on a radius  $OA$  of a circle, whose centre is  $O$ . On  $OA$  produced a point  $C$  is taken, such that  $OB \cdot OC = OA^2$ . If  $P$  be any point on the circumference of this circle,  $R$  the middle point of  $BP$ , and  $Q$  the point of intersection of  $AR$ ,  $CB$ ; prove that the locus of  $Q$  is a circle.

12. If from the middle point of a focal chord of a parabola two straight lines be drawn, one perpendicular to the chord meeting the axis in  $G$ , and the other perpendicular to the axis meeting it in  $N$ ; show that  $NG$  is constant.

13. A circle is drawn touching the axis of a parabola, the focal distance of a point  $P$ , and the diameter through  $P$ . Show that the locus of its centre is a parabola with vertex  $S$ , and latus rectum equal to  $AS$ .

14. If from the point of intersection of the directrix and axis of a parabola a chord  $XPQ$  be drawn, cutting the parabola in  $P$ ,  $Q$ ; show that the rectangle contained by the ordinates of  $PQ$  is equal to the square of one-half the latus rectum.

15. Find the locus of the centre of a circle which touches a given circle and a given straight line.

16. Given one point of contact of a parabola with three



tangents given in position, find the two other points of contact.

17. The triangle  $ABC$  circumscribes a parabola whose focus is  $S$ . Through  $A, B, C$ , lines are drawn perpendicular respectively to  $SA, SB, SC$ . Show that these lines pass through one point.

18. From the focus a line is drawn parallel to the tangent at  $P$ , meeting the parabola at  $Q$ .  $QN$  is an ordinate, and the tangents at  $P$  and  $Q$  meet the axis in  $T$  and  $T'$ . Prove that  $SN^2 = 4AT \cdot AT'$ , and that if the diameter at  $P$  meet  $SQ$  in  $E$ , the locus of  $E$  is a parabola, whose latus rectum is half that of the given parabola.

19.  $P$  is any point in a parabola; through  $S$  a line is drawn at right angles to the axis, meeting the chord  $AP$  or  $AP$  produced in  $R$ . Prove that  $SK \cdot SR = 2SY^2$ , where  $SY$  is the perpendicular on the tangent, and  $SK$  on the normal.

20. From the focus  $S$  of a parabola  $SK$  is drawn, making a given angle with the tangent at  $P$ . Show that the locus of  $K$  is that tangent to the parabola which makes with the axis an angle equal to the given angle.

21.  $PSQ$  is a focal chord of a parabola,  $AP'$  a parallel chord meeting the latus rectum in  $Q'$ ; prove that  $AP' \cdot AQ' = SP \cdot SQ$ .

22. The circle of curvature at any point of a parabola whose abscissa is  $AN$  cuts the axis in  $U$  and  $U'$ . Prove that  $AU \cdot AU' = 3AN^2$ .

23.  $AB$  is a diameter of a circle. From any point  $Q$  in the circumference a tangent  $QP$  is drawn, and from  $P$  a perpendicular  $PN$  is let fall upon  $AB$ . Show that if  $P$  be always taken so that  $QP$  is equal to  $AN$ , the locus of  $P$  will be a parabola.

24. If a tangent be drawn from any point of a parabola to the circle of curvature at the vertex, the length of the tangent

will be equal to the abscissa of the point measured along the axis.

25. To two parabolas which have a common focus and axis two tangents are drawn at right angles; the locus of their intersection is a straight line parallel to the directrices.

26. If any three tangents be drawn to a parabola, the circle described about the triangle so formed will pass through the focus, and the perpendiculars from the angles on the opposite sides intersect in the directrix.

27. A parabola touches one side of a triangle in its middle point, and the other two sides produced. Prove that the perpendicular drawn from the angles of the triangle upon any tangent to the parabola are in harmonical progression.

28. Two equal parabolas have the same axis and vertex, but are turned in opposite directions; chords of one parabola are tangents to the other. Show that the locus of the middle point of the chords is a parabola whose latus rectum is one-third of that of the given parabola.

29. Two equal parabolas have the same focus, and their axes are at right angles to each other, and a normal to one of them is perpendicular to a normal of the other; prove that the locus of the intersection of such lines is a parabola.

30. Show that in every ellipse there are two equal conjugate diameters, coinciding in direction with the diagonals of the rectangle, which touches the ellipse at the extremities of the axes.

31. If a circle be described through the two foci of an ellipse, cutting the ellipse, show that the angle between the tangents to this circle, and to the ellipse at either point of intersection, is equal to the inclination of the normal to the ellipse to the axis minor.

32. The points in which the tangents at the extremities of the transverse axis of an ellipse are cut by the tangent at any

point of the curve, are joined one with each focus; prove that the point of intersection of the joining lines lies in the normal at the point.

33. The external angle between any two tangents to an ellipse is equal to the semi-sum of the angles which the chord joining the points of contact subtends at the foci.

34. The tangent to an ellipse at any point  $P$  is cut by any two conjugate diameters in  $T, t$ , and the points  $T, t$  are joined with the foci  $S, S'$  respectively; prove that the triangles  $SPT, S'Pt$  are similar to each other.

35.  $P$  is any point on a fixed circle, the centre of which is  $O$ ;  $E$  is a fixed point without the circle; an ellipse is described with centre  $O$  and area constant so as always to touch  $EP$  at  $P$ . Find the locus of the extremities of the diameter conjugate to  $OP$ .

36. The normal at any point  $P$  of an ellipse cuts the axes in  $G, g$ ; prove that if any circle be described passing through  $G, g$ , the tangent to it from  $P$  is equal to  $CD$ .

37. Given a focus, a tangent, and the eccentricity\* of a conic section; prove that the locus of the centre is a circle.

38. A straight line is drawn through a given point  $C$  within a circle to cut it in  $P, P'$ . If  $p$  is taken in it such that  $Cp^2 = CP \cdot CP'$ , find the locus of  $p$ .

39. In the ellipse  $PY \cdot PY' : PN^2 :: CS^2 : BC^2$  and  $SY \cdot CD = SP \cdot BC$ .

40. Show that if the distance between the foci of the ellipse be greater than the length of its axis minor, there will be four positions of the tangent, for which the area of the triangle, included between it and the straight lines drawn from the centre of the curve to the feet of the perpendiculars from the foci on the tangent, will be the greatest possible.

\* The ratio of  $SA : AX$ , or of  $CS : CA$ , is called the eccentricity.

41. Two conjugate diameters of an ellipse are cut by the tangent at any point  $P$  in  $M, N$ ; prove that the area of the triangle  $CPM$  varies inversely as that of the triangle  $CPN$ .

42. Circles are described on  $SY, S'Y'$  as diameters, cutting  $SP, S'P$  respectively in  $Q, Q'$ . Prove that  $SQ \cdot S'P = SP \cdot S'Q' = BC^2$ .

43.  $PSP', pSp'$  are any two focal chords of a conic section,  $P$  and  $p$  being on the same side of the axis; prove that  $Pp, P'p'$  meet on the directrix.

44. Prove that an ellipse can be inscribed in any parallelogram so as to touch the middle points of the four sides; and show that this ellipse is the greatest of all inscribed ellipses.

45. If from any point on the exterior of two concentric, similar, and similarly placed ellipses, two tangents be drawn to the interior ellipse which also intersect the exterior; show that the distance between the points of intersection will be double of that between the points of contact.

46. The tangent at any point  $P$  in an ellipse, of which  $S$  and  $H$  are the foci, meets the axis major in  $T$ , and  $TQR$  bisects  $HP$  in  $Q$ , and meets  $SP$  in  $R$ ; prove that  $PR$  is one-fourth of the chord of curvature at  $P$  through  $S$ .

47. Prove that the distance between the two points on the circumference of an ellipse at which a given chord, not passing through the centre, subtends the greatest and least angles, is equal to the diameter which bisects that chord.

48. From any point on the auxiliary circle chords are drawn through the foci of an ellipse, and straight lines join the extremities of the chords with the extremity of the diameter passing through the point; prove that these lines will touch the ellipse.

49. A quadrilateral circumscribes an ellipse. Prove that either pair of opposite sides subtends supplementary angles at either focus.

50. Two tangents to an ellipse intersect at right angles ; show that the straight line joining their point of intersection with the point of intersection of the normals at the points of contact passes through the centre.

51.  $P, Q$  are points in two confocal ellipses, at which the line joining the common foci subtends equal angles ; prove that the tangents at  $P, Q$  are inclined at an angle which is equal to the angle subtended by  $PQ$  at either focus.

52. Tangents to an ellipse are drawn from any point in a circle through the foci ; prove that the lines bisecting the angle between the tangents, all pass through a fixed point.

53. If the ordinate at  $P$  meet the auxiliary circle in  $Q$ , and  $CQ$  meet the ellipse in  $R$ , then  $CR$  is equal to the perpendicular on the tangent at  $P$  from  $C$ .

54. If  $P$  be a point such that  $SP, S'P$  are perpendicular ; prove that  $CD^2 = 2 \cdot BC^2$ .

55. If circles be escribed to the triangle  $SPS'$  opposite to the angles  $S$  and  $S'$  ; prove that the rectangle contained by their radii is equal to  $BC^2$ .

56. The circle of curvature at any point  $P$  of an ellipse meets the focal distances in  $R, R'$  ;  $SU$  is a tangent to the circle.

Prove that  $SU^2 : SP^2 :: 2 \cdot SP - 3 \cdot AC : AC$ ,

and if  $RR'$  passes through the centre of the circle of curvature,  $CP = CS$ . Determine the limits of possibility in both cases.

57. A straight line is drawn from the centre of an ellipse meeting the ellipse in  $P$ , the circle on the major axis in  $Q$ , and the tangent at the vertex in  $T$ . Prove that as  $CT$  approaches and ultimately coincides with the semi-major axis,  $PT$  and  $QT$  are ultimately in the duplicate ratio of the axes.

58. A straight line is drawn through the focus  $S$  of an ellipse meeting the two tangents at right angles to it in  $Y$  and  $Z$ , the diameter parallel to these tangents in  $L$ , and the directrix in  $M$ ; prove that

$$SL : SY :: SZ : SM.$$

59. If any equilateral triangle  $PQR$  be described in the auxiliary circle of an ellipse, and the ordinates to  $P, Q, R$  meet the ellipse in  $P', Q', R'$ ; the circles of curvature at  $P', Q', R'$ , meet in one point lying on the ellipse.

60. From a point  $T$  two tangents  $TP, TQ$  are drawn to an ellipse. Show that a circle with  $T$  as centre can be described so as to touch  $SP, S'P, SQ, S'Q$ .

61. If the normal at  $P$  meet the axis minor in  $g$ , and if the tangent at  $P$  meet the tangent at the vertex  $A$  in  $V$ ; show that

$$Sg : SC :: PV : VA.$$

62. If a circle passing through  $Y$  and  $Y'$  touch the major axis in  $Q$ , and that diameter of the circle which passes through  $Q$  meet the tangent in  $P$ ; show that  $PQ = BC$ . (See fig. Prop. XV.)

63. If  $PG$  the normal at  $P$  cut the major axis in  $G$ , and if  $DR, PN$  be the ordinates of  $D$  and  $P$ , prove that the triangles  $PGN, DRC$  are similar; and thence deduce that  $PG$  bears a constant ratio to  $CD$ .

64. The tangent at a point  $P$  of an ellipse meets the tangents at the vertices in  $V, V'$ . On  $VV'$  as diameter, a circle is described which intersects the ellipse in  $Q, Q'$ ; show that the ordinate of  $Q$  is to the ordinate of  $P$  as  $BC$  to  $BC + CD$ ; where  $CD$  is conjugate to  $CP$ .

65.  $PCP'$  is any diameter of an ellipse; the tangents at any two points  $E$  and  $E'$  intersect in  $F$ ;  $PE', P'E$  intersect in  $G$ . Show that  $FG$  is parallel to the diameter conjugate to  $PCP'$ .

66. If  $P$  be any point on an ellipse, and with  $P$  as centre and the semi-axis minor as radius a circle be described; prove that if  $PG$  be the normal, a circle described on  $C'G$  as diameter will cut the first circle at right angles.

67.  $ABC$  is an isosceles triangle having  $AB = AC$ .  $BD, BE$  drawn on opposite sides of  $BC$ , and equally inclined to it, meet  $AC$  in  $D$  and  $E$ .

68. If an ellipse be described about  $BDC$  having its minor axis parallel to  $BC$ ; then  $AB$  will be a tangent to the ellipse.

69. If  $AQ$  be drawn from one of the vertices of an ellipse perpendicular to the tangent at any point  $P$ ; prove that the locus of the point of intersection of  $PS$  and  $QA$  produced will be a circle.

70. If  $Y, Y'$  be the feet of the perpendiculars from the foci of an ellipse on the tangent at  $P$ ; prove that the circle circumscribed about the triangle  $YNY'$  will pass through  $C$ .

71. Prove that the angle between the tangents to the auxiliary circle at  $Y, Y'$ , is the supplement of the angle  $SPS'$ .

72.  $P$  is any point on an ellipse;  $PM, PN$  perpendicular to the axes meet respectively, when produced, the circles described on the axes as diameter in the points  $Q, Q'$ . Show that  $QQ'$  passes through the centre.

73. Assuming that the greatest triangle which can be inscribed in a circle is equilateral, prove by the method of projection, that the greatest triangle which can be inscribed in an ellipse has one of its sides bisected by a diameter of the ellipse, and the others cut in points of bisection by the conjugate diameter.

74.  $PQ$  is a chord of an ellipse, normal at  $P$ ,  $LCL'$  the diameter bisecting it. Show that  $PQ$  bisects the angle  $LPL'$ , and that  $LP + L'P$  is constant.

75. A tangent to an ellipse at a point  $P$  intersects a fixed tangent in  $T$ ; if through the focus a line be drawn perpendicular to  $ST$  meeting the tangent to  $P$  in  $Q$ ; show that the locus of  $Q$  is a straight line touching the ellipse.

76. In an ellipse if a line be drawn through the focus making a constant angle with the tangent; prove that the locus of the point of intersection with the tangent is a circle.

77. Any chord  $PP'$  of an ellipse is produced to a point  $Q$ , such that  $P'Q$  is equal to half the diameter parallel to  $PP'$ , and  $QR'R$  is drawn through the centre to meet the ellipse in  $R, R'$ ; show that the area  $PCR$  is three times the area  $PCR'$ .

78. In an ellipse,  $L$  is the extremity of the latus rectum, and  $CD$  conjugate to  $CL$ . If a circle be described with centre  $C$  and passing through  $B$ , and a line be drawn through  $D$  parallel to the major axis, the portion of this line which lies within the circle will be equal to the latus rectum.

79. If  $P$  be any point in an ellipse, and  $K$  the point in which a normal at  $P$  intersects a line at right angles to it through  $S'$ ,  $E$  the point of intersection of  $SP$ , and the diameter conjugate to  $CP$ , and if  $EK$  and  $CK$  be joined, each of the figures  $SCKE$ ,  $S'CEK$  will be a parallelogram.

80. If  $T$  be a point on the axis  $AA'$  produced, and  $PN$  the ordinate of the point where the tangent from  $T$  touches the ellipse; prove that

$$AN \cdot A'N : AT \cdot A'T :: CN : CT.$$

81. Given in an ellipse a focus and two tangents; prove that the locus of the other focus is a straight line.



82. A focus, a tangent, and the axis major being given, prove that the locus of the other focus is a circle.

83. A focus, a tangent, and the axis minor being given, prove that the locus of the other focus is a straight line.

84. An ellipse touches a fixed ellipse and has a common focus with it; if the major axis be fixed, the locus of the other focus is a circle; if the minor axis is fixed, the locus is an ellipse.

85. An ellipse and a parabola have a common focus. Prove that the ellipse either intersects the parabola in two points, and has two common tangents with it, or else does not cut it.

86. If in the ellipse a focus, a point, and the axis minor be given, the locus of the other focus is a parabola.

87. If at the extremities  $P, Q$  of any two diameters  $CP, CQ$  of an ellipse, two tangents  $Pp, Qq$  be drawn cutting each other in  $T$ , and the diameter produced in  $p$  and  $q$ , then the areas of the triangles  $TQp, TPq$  are equal.

88. If a straight line  $CN$  be drawn from the centre to bisect that chord of the circle of curvature at any point  $P$  of an ellipse, which is common to the ellipse and circle, and if it be produced to cut the ellipse in  $Q$ , and the tangent in  $T$ ; prove that  $CP = CQ$ , and that each is a mean proportional between  $CN$  and  $CT$ .

89. A parabola of given latus rectum is described touching symmetrically two conjugate diameters of an ellipse; find the locus of the focus.

90. An ellipse is described so as to touch the three sides of a triangle; prove that if one of its foci move along the circumference of a circle passing through two of the angular points of the triangle, the other will move along the circumference of another circle, passing through the same two angular points. Prove also that if one of these circles pass through the centre of the circle inscribed in the triangle, the two circles will coincide.

91. A triangle is described about an ellipse, so that the extremities of one of its sides lie in an ellipse, confocal with the given one; prove that the line bisecting the opposite angle passes through the pole of that side with respect to the outer ellipse.

92. Prove the following construction for a pair of tangents from any external point  $T$  to an ellipse of which the centre is  $C$ . Join  $CT$ ; let  $TPCP'T$ , a similar and similarly situated ellipse, be drawn, of which  $CT$  is a diameter, and  $P, P'$  its points of intersection with the given ellipse;  $TP, TP'$  will be tangents to the given ellipse.

93. The locus of the foci of all ellipses inscribed in the same parallelogram is a rectangular hyperbola. Prove this, and give a geometrical construction for finding the asymptotes.

94.  $AC$  is a fixed diameter of a circle,  $ABCD$  a quadrilateral figure inscribed in the circle; prove that if the angles  $BAC, DAC$  be complementary, the locus of the intersection of  $BA, CD$  will be an hyperbola.

95. Prove that a circle can be described so as to touch the four straight lines drawn from the foci of an hyperbola to any two points on the same branch of the curve.

96. Any three diameters of an ellipse  $LL', MM', NN'$ , being taken, a circumscribing parallelogram  $RTUV$  touches the ellipse at  $L, L', M, M'$ . Show that a conic section can be described through the points  $R, T, U, V, N, N'$ , which will be an hyperbola whose asymptotes are the lines forming in the ellipse the diameters conjugate to  $NN'$  and to the other common chord of the ellipse and hyperbola.

97. On opposite angles of any chord of a rectangular hyperbola are described equal segments of circles; show that the four points in which the circles to which these segments belong again meet the hyperbola, are the angular points of a parallelogram.

98. A triangle is inscribed in a rectangular hyperbola; prove that the circle described through the middle points of the sides of the triangle passes through the centre of the hyperbola.

99.  $ACB$  is an isosceles triangle;  $AB$  the base, and  $D$  any point in  $CB$  or  $CB$  produced: if  $BZ$  be drawn parallel to  $AD$ , meeting  $CA$  or  $CA$  produced in  $Z$ , prove that the middle point of  $DZ$  will be in an hyperbola whose asymptotes are  $CA$ ,  $CB$ .

100. An ellipse and hyperbola are described so that the foci of each are at the extremities of the transverse axis of the other; prove that the tangents at their points of intersection meet the conjugate axis in points equidistant from the centre.

101. In a rectangular hyperbola,  $PK$ ,  $PL$  are drawn at right angles to  $A'P$ ,  $AP$  respectively, to meet the transverse axis in  $K$  and  $L$ ; prove that  $PK$  is equal to  $AP$  and  $KL$  to  $AA'$ , and the normal at  $P$  bisects  $KL$ .

102. In a rectangular hyperbola  $PO$  is a fixed diameter,  $Q$  any point on the curve; show that the angles  $QPO$ ,  $QOP$  differ by a constant angle.

103. If the tangent at any point  $P$  of an hyperbola cut an asymptote in  $T$ , and if  $SP$  cut the same asymptote in  $Q$ , then  $SQ = QT$ .

104. If a given point be the focus of any hyperbola, passing through a given point and touching a given straight line, prove that the locus of the other focus is an arc of a fixed hyperbola.

105. If a series of conics be described having equal latera recta, the focus of a given parabola as their common focus, and the tangents to the parabola as their directrices; the common tangents to any two of the series will intersect in the directrix of the parabola.

106. At any  $P$  of an hyperbola a tangent is drawn, and  $PQ$  is taken on it in a constant ratio to  $CD$ ; prove that the locus of  $Q$  is an hyperbola.

107. If an hyperbola be described touching the four sides of a quadrilateral inscribed in a circle, and one focus lie on a circle, the other focus will also lie on this circle.

108. In an hyperbola, supposing the two asymptotes and one point of the curve to be given in position, show how to construct the curve; and find the position of the foci.

109. If  $A, D$  be two fixed points, and the angle  $PAD$  always exceed  $PDA$  by a given angle; find the locus of  $P$ , and the position of the transverse axis and asymptote.

110. From the middle point  $D$  of the base  $AB$  of the triangle  $ABC$  a straight line  $EDE'$  is drawn, making a given angle with  $AB$ , and the points  $E, E'$  are taken so that  $ED = E'D = \frac{1}{2} AB$ . If  $CA, CB$  take all possible positions consistent with the condition that the difference of the angles  $CAB, CBA$  is equal to  $EDA$ ; prove that the point  $C$  will trace out a rectangular hyperbola of which  $AB, EE'$  are conjugate diameters.

111. In the rectangular hyperbola, prove that the triangle, formed by the tangent at any point and its intercepts on the axes, is similar to the triangle formed by the straight line joining that point with the centre, and the abscissa and semi-ordinate of the point.

112. Tangents are drawn to an hyperbola, and the portion of each tangent intercepted by the asymptotes is divided in a constant ratio; prove that the locus of the point of section is an hyperbola.

113. Show that the point of trisection of a series of continuous circular arcs lie on branches of two hyperbolas and determine the distance between their centres.

114. From a point  $R$  on one asymptote  $RE$  is drawn touching the hyperbola in  $E$ , and  $ET$ ,  $EV$  are drawn through  $E$ , parallel to the asymptotes, cutting a diameter in  $T$  and  $V$ ;  $RV$  is joined, cutting the hyperbola in  $P$ ,  $p$ : show that  $TP$ ,  $Tp$  touch the hyperbola.

115. Given in the ellipse a focus and two points, the locus of the other focus is an hyperbola.

116. If a rectangular hyperbola passes through three given points, the locus of its centre is a circle, which passes through the middle points of the lines joining the three given points.

117. If the tangent at  $P$  meet one asymptote in  $T$ , and a line  $TQ$  be drawn parallel to the other asymptote to meet the curve in  $Q$ ; prove that if  $PQ$  be joined and produced both ways to meet the asymptotes in  $R$  and  $R'$ ,  $RR'$  will be trisected at the points  $P$  and  $Q$ .

118. If two concentric rectangular hyperbolas have a common tangent, the lines joining their points of intersection to their respective points of contact with the common tangent will subtend equal angles at their common centres.

119. If  $TP$ ,  $TQ$  be two tangents drawn from any point  $T$  to touch a conic in  $P$  and  $Q$ , and if  $S$  and  $S'$  be the foci, then

$$ST^2 : S'T^2 :: SP \cdot SQ : S'P \cdot S'Q.$$

120. The circle of curvature at the vertex of a conic meets the axis again in  $D$ , and a tangent is drawn to the circle at  $D$ : if two tangents be drawn to the circle from any point in the conic they will intercept between them a constant length of the former tangent.

121. A family of hyperbolas is described about a given triangle; prove that if one of the asymptotes always passes through a fixed point the other will always touch a fixed conic section to which the three sides of the triangle are tangents.

122. If the lines which bisect the angles between pairs of tangents to an ellipse be parallel to a fixed straight line, prove that the locus of the points of intersection of the tangents will be a rectangular hyperbola.

123. An hyperbola, of given eccentricity, always passes through two given points; if one of its asymptotes always pass through a third given point in the same straight line with these, prove that the locus of the centre of the hyperbola will be a circle.

124.  $A, P$  and  $B, Q$  are points taken respectively in two parallel straight lines,  $A$  and  $B$  being fixed, and  $P, Q$  variable. Prove that if the rectangle  $APBQ$  be constant, the line  $PQ$  will always touch a fixed ellipse or a fixed hyperbola, according as  $P$  and  $Q$  are on the same or opposite sides of  $AB$ .

125. If two plane sections of a right cone be taken, having the same directrix, the foci corresponding to that directrix lie on a straight line which passes through the vertex.

126. Give a geometrical construction by which a cone may be cut so that the section may be an ellipse of given eccentricity.

127. Given a right cone and a point within it, there are but two sections which have this point for focus; and the planes of these sections make equal angles with the straight line joining the given point and the vertex of the cone.

128. If the curve formed by the intersection of any plane with a cone be projected upon a plane perpendicular to the axis, prove that the curve of projection will be a conic section, having its focus at the point in which the axis meets the plane of projection.

129. If  $F$  be the point where the major axis of an elliptic

section meets the axis of the cone, and  $C$  be the centre of the section; prove that

$$CF : CS :: AA' : AO + A'O,$$

$O$  being the vertex of the cone.

130. Two intersecting right cones have the same axis, and the vertical angle of the interior is equal to that of an equilateral triangle; show that the vertex of the interior cone is a focus of all sections of the exterior made by planes touching the interior.

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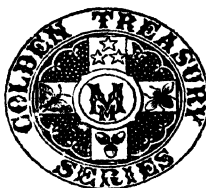
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